# ALGEBRA QUALIFYING EXAM SPRING 2012

Work all of the problems. *Justify the statements in your solutions by reference to specific results*, as appropriate. Partial credit is awarded for partial solutions. The set of rational numbers is $\boldsymbol{Q}$ and set of the complex numbers is $\boldsymbol{C}$.

1. Let $I$ be an ideal of $R = \boldsymbol{C}[x_1, \ldots, x_n]$. Show that $\dim_{\boldsymbol{C}} R/I$ is finite $\Leftrightarrow$ $I$ is contained in only finitely many maximal ideals of $R$.

2. If $G$ is a group with $|G| = 7^2 \cdot 11^2 \cdot 19$, show that $G$ must be abelian and describe the possible structures of $G$.

3. Let $F$ be a finite field and $G$ a finite group with GCD$\{\mathit{char}\ F, |G|\} = 1$. The group algebra $F[G]$ is an algebra over $F$ with $G$ as an $F$-basis, elements $\alpha = \sum_G a_g g$ for $a_g \in F$, and multiplication that extends $ag \cdot bh = ab \cdot gh$. Show that any $x \in F[G]$ that is not a zero left divisor (i.e. if $xy = 0$ for $y \in F[G]$ then $y = 0$) must be invertible in $F[G]$.

4. If $p(x) = x^8 + 2x^6 + 3x^4 + 2x^2 + 1 \in \boldsymbol{Q}[x]$ and if $\boldsymbol{Q} \subseteq M \subseteq \boldsymbol{C}$ is a splitting field for $p(x)$ over $\boldsymbol{Q}$, argue that Gal$(M/\boldsymbol{Q})$ is solvable.

5. Let $R$ be a commutative ring with 1 and let $x_1, \ldots, x_n \in R$ so that $x_1y_1 + \cdots + x_ny_n = 1$ for some $y_j \in R$. Let $A = \{(r_1, \ldots, r_n) \in R^n \mid x_1r_1 + \cdots + x_nr_n = 0\}$. Show that $R^n \cong_R A \oplus R$, that $A$ has $n$ generators, and that when $R = F[x]$ for $F$ a field then $A_R$ is free of rank $n - 1$.

6. For $p$ a prime let $F_p$ be the field of $p$ elements and $K$ an extension field of $F_p$ of dimension 72.
   *i*) Describe the possible structures of $Gal(K/F_p)$.
   *ii*) If $g(x) \in F_p[x]$ is irreducible of degree 72, argue that $K$ is a splitting field of $g(x)$ over $F_p$.
   *iii*) Which integers $d > 0$ have irreducibles in $F_p[x]$ of degree $d$ that split in $K$?

Algebra Graduate Exam

Fall 2012

Work all the problems. Be as explicit as possible in your solutions, and justify your statements with specific reference to the results that you use. Partial credit will be given for partial solutions.

1. Use Sylow's theorems directly to find, up to isomorphism, all possible structures of groups of order $5{\cdot}7{\cdot}23$.

2. Let $A$, $B$, and $C$ be finitely generated $F[x] = R$ modules, for $F$ a field, with $C$ torsion free. Show that $A \square_R C \square B \square_R C$ implies that $A \square B$. Show by example that this conclusion can fail when $C$ is not torsion free.

3. Working in the polynomial ring $\boldsymbol{C}[x, y]$, show that some power of $(x + y)(x^2 + y^4 - 2)$ is in $(x^3 + y^2, y^3 + xy)$.

4. For integers $n$, $m$ >1, let $A \subseteq M_n(\boldsymbol{Z}_m)$ be a subring with the property that if $x \in A$ with $x^2 = 0$ then $x = 0$. Show that $A$ is commutative. Is the converse true?

5. Let F be the splitting field of $f(x) = x^6 - 2$ over $\mathbf{Q}$. Show that $\mathrm{Gal}(F/\mathbf{Q})$ is isomorphic to the dihedral group of order 12.

6. Given that all groups of order 12 are solvable show that any group of order $2^2{\cdot}3{\cdot}7^2$ is solvable.

Algebra Graduate Exam

Spring 2013

Work all the problems. Be as explicit as possible in your solutions, and justify your statements with specific reference to the results that you use. Partial credit will be given for partial solutions.

1. Let $p > 2$ be a prime. Describe, up to isomorphism, all groups of order $2p^2$.

2. Let R be a commutative Noetherian ring with 1. Show that every proper ideal of R is the product of finitely many (not necessarily distinct) prime ideals of R.
(Hint: Consider the set of ideals that are not products of finitely many prime ideals. Also, note that if R is not a prime ring then $IJ=(0)$ for some non-zero ideals I and J of R)

3. In the polynomial ring $R = \mathbf{C}[x,y,z]$ show that there is a positive integer m, and polynomials $f, g, h \in R$ such that
$$(x^{16}y^{25}z^{81} - x^7z^{15} - yz^9 + x^5)^m = (x - y)^3 f + (y - z)^5 g + (x + y + z - 3)^7 h.$$

4. Let $R \neq (0)$ be a finite ring such that for any $x \in R$ there is $y \in R$ with $xyx = x$. Show that R contains an identity element and that, for $a, b \in R$, if $ab=1$ then $ba=1$.

5. Let $f(x) = x^{15} - 2$, and let L be the splitting field of $f(x)$ over $\mathbf{Q}$.
a) What is $[L:\mathbf{Q}]$?
b) Show there exists a subfield F of degree 8 that is Galois over $\mathbf{Q}$.
c) What is $\mathrm{Gal}(F/\mathbf{Q})$?
d) Show there is a subgroup of $\mathrm{Gal}(L/\mathbf{Q})$ that is isomorphic to $\mathrm{Gal}(F/\mathbf{Q})$.

6. Let $F/\mathbf{Q}$ be a Galois extension of degree 60, and suppose F contains a primitive ninth root of unity. Show $\mathrm{Gal}(F/\mathbf{Q})$ is solvable.

7. Let n be a positive integer. Show that $f(x,y) = x^n + y^n + 1$ is irreducible in $\mathbf{C}[x,y]$.

# Algebra Qualifying Exam - Fall 2013

1. Let $H$ be a subgroup of the symmetric group $S_5$. Can the order of $H$ be $15$, $20$ or $30$?

2. Let $R$ be a PID and $M$ a finitely generated torsion module of $R$. Show that $M$ is a cyclic $R$-module if and only if for any prime $\mathfrak{p}$ of $R$ either $\mathfrak{p}M = M$ or $M/\mathfrak{p}M$ is a cyclic $R$-module.

3. Let $R = \mathbb{C}[x_1, \ldots, x_n]$ and suppose $I$ is a proper non-zero ideal of $R$. The coefficients of a matrix $A \in M_n(R)$ are polynomials in $x_1, \ldots, x_n$ and can be evaluated at $\beta \in \mathbb{C}^n$; write $A(\beta) \in M_n(\mathbb{C})$ for the matrix so obtained. If for some $A \in M_n(R)$ and all $\alpha \in Var(I)$, $A(\alpha) = 0_{n\times n}$, show that for some integer $m$, $A^m \in M_n(I)$.

4. If $R$ is a noetherian unital ring, show that the power series ring $R[[x]]$ is also a noetherian unital ring.

5. Let $p$ be a prime. Prove that $f(x) = x^p - x - 1$ is irreducible over $\mathbb{Z}/p\mathbb{Z}$. What is the Galois group? (Hint: observe that if $\alpha$ is a root of $f(x)$, then so is $\alpha + i$ for $i \in \mathbb{Z}/p\mathbb{Z}$.)

6. Let $R$ be a finite ring with no nilpotent elements. Show that $R$ is a direct product of fields.

7. Let $K \subset \mathbb{C}$ be the field obtained by adjoining all roots of unity in $\mathbb{C}$ to $\mathbb{Q}$. Suppose $p_1 < p_2$ are primes, $a \in \mathbb{C} \setminus K$, and write $L$ for a splitting field of
$$g(x) = (x^{p_1} - a)(x^{p_2} - a)$$
over $K$. Assuming each factor of $g(x)$ is irreducible, determine the order and the structure of $Gal(L/K)$.

# ALGEBRA QUALIFYING EXAM SPRING 2014

Work all of the problems. Justify the statements in your solutions by reference to specific results, as appropriate. Partial credit is awarded for partial solutions. The set of rational numbers is $\boldsymbol{Q}$, and set of the complex numbers is $\boldsymbol{C}$. *Hand in solutions in order of the problem numbers.*

1. Let $L$ be a Galois extension of a field $F$ with $\mathrm{Gal}(L/F) \cong D_{10}$, the dihedral group of order 10. How many subfields $F \subseteq M \subseteq L$ are there, what are their dimensions over $F$, and how many are Galois over $F$?

2. Up to isomorphism, using direct and semi-direct products, describe the possible structures of a group of order $5 \cdot 11 \cdot 61$.

3. Let $I$ be a nonzero ideal of $R = \boldsymbol{C}[x_1, \ldots, x_n]$. Show that $R/I$ is a finite dimensional algebra over $\boldsymbol{C}$ if and only if $I$ is contained in only finitely many maximal ideals of $R$.

4. Let $R$ be a commutative ring with 1, and $M$ a noetherian $R$ module. For $N$ a noetherian $R$ module show that $M \otimes_R N$ is a noetherian $R$ module. When $N$ is an artinian $R$ module show that $M \otimes_R N$ is an artinian $R$ module.

5. For $n \geq 5$ show that the symmetric group $S_n$ cannot have a subgroup $H$ with $3 \leq [S_n : H] < n$ ($[S_n : H]$ is the index of $H$ in $S_n$).

6. Let $R$ be the group algebra $\boldsymbol{C}[S_3]$. How many nonisomorphic, irreducible, left modules does $R$ have and why?

7. Let each of $g_1(x), g_2(x), \ldots, g_n(x) \in \boldsymbol{Q}[x]$ be irreducible of degree four and let $L$ be a splitting field over $\boldsymbol{Q}$ for $\{g_1(x), \ldots, g_n(x)\}$. Show there is an extension field $M$ of $L$ that is a radical extension of $\boldsymbol{Q}$.

**Algebra Exam September 2014**

Show your work. Be as clear as possible. Do all problems.
Hand in solutions in numerical order.

1. Let $G$ be a group of order 56 having at least 7 elements of order 7. Let $S$ be a Sylow 2-subgroup of $G$.

   (a) Prove that $S$ is normal in $G$ and $S = C_G(S)$.

   (b) Describe the possible structures of $G$ up to isomorphism. (Hint: How does an element of order 7 act on the elements of $S$?)

2. Show that a finite ring with no nonzero nilpotent elements is commutative.

3. If $R = M_n(\mathbb{Z})$, and $A$ is an additive subgroup of $R$, show that as additive subgroups $[R : A]$ is finite if and only if $R \otimes_{\mathbb{Z}} \mathbb{Q} = A \otimes_{\mathbb{Z}} \mathbb{Q}$.

4. Let $R$ be a commutative ring with 1, $n$ a positive integer and $A_1, \ldots A_k \in M_n(R)$. Show that there is a noetherian subring $S$ of $R$ containing 1 with all the $A_i \in M_n(S)$.

5. Let $R = \mathbb{C}[x, y]$. Show that there exists a positive integer $m$ such that $((x + y)(x^2 + y^4 - 2))^m$ is in the ideal $(x^3 + y^2, y^3 + xy)$.

6. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree $n \geq 5$. Let $L$ be the splitting field of $f$ and let $\alpha \in L$ be a zero of $f$. Given that $[L : Q] = n!$, prove that $\mathbb{Q}]\alpha^4] = \mathbb{Q}[\alpha]$.

# ALGEBRA QUALIFYING EXAM FALL 2015

Work all of the problems. *Justify the statements in your solutions by reference to specific results, as appropriate.* Partial credit is awarded for partial solutions. The set of integers is $\boldsymbol{Z}$, the set of rational numbers is $\boldsymbol{Q}$, and set of the complex numbers is $\boldsymbol{C}$.

Hand in the exam with problems in numerical order.

1. If $M$ is a maximal ideal in $\boldsymbol{Q}[x_1, \ldots, x_n]$ show that there are only finitely many maximal ideals in $\boldsymbol{C}[x_1, \ldots, x_n]$ that contain $M$.

2. Let $R$ be a right Noetherian ring with 1. Prove that $R$ has a *unique* maximal nilpotent ideal $P(R)$. Argue that $R[x]$ also has a unique maximal nilpotent ideal $P(R[x])$. Show that $P(R[x]) = P(R)[x]$.

3. Up to isomorphism, describe the possible structures of any group of order 182 as a direct sum of cyclic groups, dihedral groups, other semi-direct products, symmetric groups, or matrix groups. (Note: 91 is not a prime!)

4. Let $K = \boldsymbol{C}(y)$ for an indeterminate $y$ and let $p_1 < p_2 < \cdot\cdot\cdot < p_n$ be primes (in $\boldsymbol{Z}$). Let $f(x) = (x^{p_1} - y)\cdots(x^{p_n} - y) \in K$ with splitting field $L$ over $K$

   a) Show each $x^{p_j} - y$ is irreducible over $K$.

   b) Describe the structure of $Gal(L/K)$.

   c) How many intermediate fields are between $K$ and $L$?

5. In any finite ring $R$ with 1 show that some element in $R$ is not a sum of nilpotent elements. Note that in all $M_n(\boldsymbol{Z}/n\boldsymbol{Z})$ the identity matrix is a sum of nilpotent elements. (Hint: What is the trace of a nilpotent element in a matrix ring over a field?)

6. Let $R$ be a commutative principal ideal domain.

   (1) If $I$ and $J$ are ideals of $R$, show $R/I \otimes_R R/J \cong R/(I + J)$.

   (2) If $V$ and $W$ are finitely generated $R$ modules so that $V \otimes_R W = 0$, show that $V$ and $W$ are torsion modules whose annihilators in $R$ are relatively prime.

7. Let $g(x) = x^{12} + 5x^6 - 2x^3 + 17 \in \boldsymbol{Q}[x]$ and $F$ a splitting field of $g(x)$ over $\boldsymbol{Q}$. Determine if $Gal(F/\boldsymbol{Q})$ is solvable.

# Algebra Qualifying Exam - Fall 2016

1. If $R := \mathbb{C}[x, y]/(y^2 - x^3 - 1)$, then describe all the maximal ideals in $R$.

2. Suppose $F$ is a field, and $\mathfrak{b}_n(F)$ is the $F$-algebra of upper-triangular matrices, i.e., the subalgebra of $M_n(F)$ consisting of matrices $X$ such that $X_{ij} = 0$ when $i > j$. Describe the Jacobson radical of $\mathfrak{b}_n(F)$, the simple modules, and the maximal semi-simple quotient.

3. Let $\mathbb{F}_5$ be the finite field with 5 elements, and consider the group $G = PGL_2(\mathbb{F}_5)$ (i.e., the quotient of the group of invertible $2 \times 2$-matrices over $\mathbb{F}_5$ by the subgroup of scalar multiples of the identity).
   (a) What is the order of $G$?
   (b) Describe $N_G(P)$ where $P$ is a Sylow-5 subgroup of $G$.
   (c) If $H \subset G$ is a subgroup, can $H$ have order 15, 20 or 30?

4. Let $A$ be an $n \times n$ matrix over $\mathbb{Z}$. Let $V$ be the $\mathbb{Z}$-module of column vectors of size $n$ over $\mathbb{Z}$.
   (a) Prove that the size of $V/AV$ is equal to the absolute value of $\det(A)$ if $\det(A) \neq 0$.
   (b) Prove that $V/AV$ is infinite if $\det(A) = 0$.

   (hint: use the theory of finitely generated modules $\mathbb{Z}$-modules)

5. Let $V$ be a finite dimensional right module over a division ring $D$. Let $W$ be a $D$-submodule of $V$.
   (a) Let $I(W) = \{f \in \mathrm{End}_D(V) | f(W) = 0\}$. Prove that $I(W)$ is a left ideal of $\mathrm{End}(V)$.
   (b) Prove that any left ideal of $\mathrm{End}_D(V)$ is $I(W)$ for some submodule $W$.

6. Let $p$ and $q$ be distinct primes. Let $F$ be the subfield of $\mathbb{C}$ generated by the $pq$-roots of unity. Let $a, b$ be squarefree integers all greater than 1. Let $c, d \in \mathbb{C}$ with $c^p = a$ and $d^q = b$. Let $K = F(c, d)$.
   (a) Show that $K/\mathbb{Q}$ is a Galois extension.
   (b) Describe the Galois group of $K/F$.
   (c) Show that any intermediate field $F \subset L \subset K$ satisfies $L = F(S)$ where $S$ is some subset of $\{c, d\}$.

# Algebra Exam February 2015

Show your work. Be as clear as possible. Do all problems.

1. Use Sylow's theorems and other results to describe, up to isomorphism, the possible structures of a group of order 1005.

2. Let $R$ be a commutative ring with 1. Let $M, N$ and $V$ be $R$-modules.

   (a) Show if that $M$ and $N$ are projective, then so is $M \otimes_R N$.

   (b) Let $\operatorname{Tr}(V) := \{\sum_i \phi_i(v_i) | \phi \in \operatorname{Hom}_R(V, R), v_i \in V\} \subset R$. If $1 \in \operatorname{Tr}(V)$, show that up to isomorphism $R$ is a direct summand of $V^k$ for some $k$.

3. Let $F$ be a field and $M$ a maximal ideal of $F[x_1, \ldots, x_n]$. Let $K$ be an algebraic closure of $F$. Show that $M$ is contained in at least 1 and in only finitely many maximal ideals of $K[x_1, \ldots, x_n]$.

4. Let $F$ be a finite field.

   (a) Show that there are irreducible polynomials over $F$ of every positive degree.

   (b) Show that $x^4 + 1$ is irreducible over Q$[x]$ but is reducible over $\mathbb{F}_p[x]$ for every prime $p$ (hint: show there is a root in $\mathbb{F}_{p^2}$).

5. Let $F$ be a field and $M$ a finitely generated $F[x]$-module. Show that $M$ is artiniian if and only if $\dim_F M$ is finite.

6. Let $R$ be a right Artinian ring with with a faithful irreducible right $R$-module. If $x, y \in R$, set $[x, y] := xy - yx$. Show that if $[[x, y], z] = 0$ for all $x, y, z \in R$, then $R$ has no nilpotent elements.

**Algebra Exam February 2016**

Show your work. Be as clear as possible. Do all problems.

1. Let $R$ be a Noetherian commutative ring with 1 and $I \neq 0$ an ideal of $R$. Show that there exist finitely many nonzero prime ideals $P_i$ of $R$ (not necessarily distinct) so that $\Pi_i P_i \subset I$ (Hint: consider the set of ideals which are not of that form).

2. Describe all groups of order 130: show that every such group is isomorphic to a direct sum of dihedral and cyclic groups of suitable orders.

3. Let $f(x) = x^{12} + 2x^6 - 2x^3 + 2 \in \mathbb{Q}[x]$. Show $f(x)$ is irreducible. Let $K$ be the splitting field of $f(x)$ over $\mathbb{Q}$. Determine whether $\mathrm{Gal}(K/Q)$ is solvable.

4. Determine up to isomorphism the algebra structure of $\mathbb{C}[G]$ where $G = S_3$ is the symmetric group of degree 3 (Recall that $\mathbb{C}[G]$ is the group algebra of $G$ which has basis $G$ and the multiplication comes from the multiplication on $G$).

5. If $F$ is a field and $n > 1$ show that for any nonconstant $g \in F[x_1, \ldots, x_n]$ the ideal $gF[x_1, \ldots, x_n]$ is not a maximal ideal of $F[x_1, \ldots, x_n]$.

6. Let $F$ be a field and let $P$ be a submodule of $F[x]^n$. Suppose that the quotient module $M := F[x]^n/P$ is Artinian. Show that $M$ is finite dimensional over $F$.

**Algebra Exam January 2017**

Show your work. Be as clear as possible. Do all problems.

1. Let $R$ be a PID. Let $M$ be an $R$-module.
   (a) Show that if $M$ is finitely generated, then $M$ is cyclic if and only if $M/PM$ is for all prime ideals $P$ of $R$.
   (b) Show that the previous statement is false if $M$ is not finitely generated.

2. Prove that a power of the polynomial $(x+y)(x^2+y^4-2)$ belongs to the ideal $(x^3+y^2, x^3+xy)$ in $\mathbb{C}[x,y]$.

3. Let $G$ be a finite group with a cyclic Sylow 2-subgroup $S$.
   (a) Show that $N_G(S) = C_G(S)$.
   (b) Show that if $S \neq 1$, then $G$ contains a normal subgroup of index 2 (hint: suppose that $n = [G : S]$, consider an appropriate homomorphism from $G$ to $S_n$).
   (c) Show that $G$ has a normal subgroup $N$ of odd order such that $G = NS$.

4. Show that $\mathbb{Z}[\sqrt{5}]$ is not integrally closed in its quotient field.

5. Let $f(x) = x^{11} - 5 \in \mathbb{Q}[x]$.
   (a) Show that $f$ is irreducible in $\mathbb{Q}[x]$.
   (b) Let $K$ be the splitting field of $f$ over $\mathbb{Q}$. What is the Galois group of $K/\mathbb{Q}$.
   (c) How many subfields $L$ of $K$ are there so such that $[K : L] = 11$.

6. Suppose that $R$ is a finite ring with 1 such that every unit of $R$ has order dividing 24. Classify all such $R$.

# Algebra Qualifying Exam - Fall 2017

1. Assume $S$ is a commutative integral domain, and $R \subset S$ is a subring. Assume $S$ is finitely generated as an $R$-module, i.e., there exist elements $s_1, \ldots, s_n \in S$ such that $S = s_1R + s_2R + \cdots + s_nR$. Show that $R$ is a field if and only if $S$ is a field. Is the statement true if the assumption that $S$ is an integral domain is dropped?

2. Suppose $R$ is a commutative unital ring, $\mathfrak{p} \subset R$ is a prime ideal and $M$ is a finitely generated $R$-module. Recall that the annihilator ideal $\mathrm{Ann}_R(M)$ consists of elements $r \in R$ such that $rm = 0$ for all $m \in M$. Show the localized module $M_{\mathfrak{p}}$ is *non-zero* if and only if $\mathrm{Ann}_R(M) \subset \mathfrak{p}$.

3. Let $f(x) = x^5 + 1$. Describe the splitting field $K$ of $f(x)$ over $\mathbb{Q}$ and compute the Galois group $Gal(K/\mathbb{Q})$.

4. Let $\alpha$ be the real positive 16th root of 3 and consider the field $F = \mathbb{Q}(\alpha)$ generated by $\alpha$ over the field of rational numbers. Observe that there is a chain of intermediate fields
$$\mathbb{Q} \subset \mathbb{Q}(\alpha^8) \subset \mathbb{Q}(\alpha^4) \subset \mathbb{Q}(\alpha^2) \subset \mathbb{Q}(\alpha) = F.$$
Compute the degrees of these intermediate field extensions and conclude they are all distinct. Show that every intermediate field $K$ between $\mathbb{Q}$ and $F$ is one of the above (hint: consider the constant term of the minimal polynomial of $\alpha$ over $K$)

5. A finite group is said to be *perfect* if it has no nontrivial abelian homomorphic image. Show that a perfect group has no non-trivial solvable homomorphic image. Next, suppose that $H \subset G$ is a normal subgroup with $G/H$ perfect. If $\theta : G \to S$ is a homomorphism from $G$ to a solvable group $S$ and if $N = \ker \theta$, show that $G = NH$ and deduce that $\theta(H) = \theta(G)$.

6. Let $A$ be a finite-dimensional $\mathbb{C}$-algebra. Given $a \in A$, write $\mathrm{L}_a$ for the left-multiplication operator, i.e., $\mathrm{L}_a(b) = ab$. Define a map $(-,-) : A \times A \to \mathbb{C}$ by means of the formula $(a, b) := \mathrm{Tr}\, \mathrm{L}_a \, \mathrm{L}_b$.
   (a) Show that $(-,-)$ is a symmetric bilinear form on $A$.
   (b) If one defines the radical $\mathrm{Rad}(-,-)$ as $\{a \in A | (a, b) = 0 \ \forall b \in A\}$, then show that $\mathrm{Rad}(-,-)$ is a two-sided ideal in $A$.
   (c) Show that $\mathrm{Rad}(-,-)$ coincides with the Jacobson radical of $A$.

7. Suppose $F$ is an algebraically closed field, $V$ is a finite-dimensional $F$-vector space, and $A \in \mathrm{End}_F(V)$. Show that there exist polynomials $f, g \in F[x]$ such that i) $A = f(A) + g(A)$, ii) $f(A)$ is diagonalizable and $g(A)$ nilpotent, and iii) $f$ and $g$ both vanish at $0$.

# Algebra Qualifying Exam - Spring 2018

1. Prove that a group of order $72$ cannot be simple.

2. Say that a group $G$ is uniquely $p$-divisible if the $p$-th power map sending $x \in G$ to $x^p$ is bijective. Show that if $G$ is a finitely generated uniquely $p$-divisible *abelian* group, then $G$ is finite and has order coprime to $p$.

3. Let $\mathbb{Q}$ be the field of rational numbers and consider $f(x) = x^8 + x^4 + 1 \in \mathbb{Q}[x]$. Write $E$ for a splitting field for $f(x)$ over $\mathbb{Q}$ and set $G = Gal(E/\mathbb{Q})$. Find $|E : \mathbb{Q}|$ and determine the Galois group $G$ up to isomorphism. If $\Omega \subset E$ is the set of roots of $f(x)$, find the number of orbits for the action of $G$ on $\Omega$.

4. Show that a $10$-dimensional $\mathbb{C}$-algebra necessarily contains a non-zero nilpotent element (hint: what can you say about the Jacobson radical of such an algebra?).

5. Consider the algebra $A := \mathbb{C}[M_n(\mathbb{C})]$ of polynomial functions on the ring of $n \times n$ complex matrices $M_n(\mathbb{C})$. Consider the polynomial functions defined by the formula $P_{ij}(X) := (X^n)_{ij}$. Let $I \subset A$ be the ideal defined by $P_{ij}$, $1 \leq i, j \leq n$. Describe the variety $V(I)$ and use your description to show that $I \neq \sqrt{I}$.

6. Is the ring $k[x, y]/(y^2 - x^3)$ integrally closed in its field of fractions?

7. Suppose $R$ is a commutative (unital) ring, $M$ and $N$ are $R$-modules and $f : M \to N$ is an $R$-module homomorphism. Show that $f$ is surjective if and only if, for every prime ideal $\mathfrak{p} \subset R$, the induced map $f_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ of modules over $R_{\mathfrak{p}}$ is surjective.

# Algebra Qualifying Exam - Fall 2018

1. Let $\mathbb{F}_p$ be a finite field with $p$ elements, and consider the group $GL_n(\mathbb{F}_p)$. Write down the order of $GL_n(\mathbb{F}_p)$ and a Sylow $p$-subgroup.

2. Prove that there are no simple groups of order $600$.

3. Prove that $\mathbb{Z}[\sqrt{10}]$ is integrally closed in its field of fractions, but not a UFD.

4. If $F$ is a field and $E/F$ is an extension, then an element $a \in E$ will be called abelian if $Gal(F[a]/F)$ is an abelian group. Show that the set of abelian elements of $E$ is a subfield of $E$ containing $F$.

5. Let $K$ be the splitting field of $x^4 - 2 \in \mathbb{Q}[x]$. Prove that $Gal(K/\mathbb{Q})$ is $D_8$ the dihedral group of order $8$ (i.e., the group of isometries of the square). Find all subfields of $K$ that have degree $2$ over $\mathbb{Q}$.

6. Let $F$ be a field, and suppose $A$ is a finite-dimensional $F$-algebra. Write $[A, A]$ for the $F$-subspace of $A$ spanned by elements of the form $ab - ba$ with $a, b \in A$. Show that $[A, A] \neq A$ in the following two cases:

   (a) when $A$ is a matrix algebra over $F$;

   (b) when $A$ is a central division algebra over $F$.

   (Recall that a division algebra over $F$ is called central if its center is isomorphic with $F$.)

7. If $\varphi : A \to B$ is a surjective homomorphism of rings, show that the image of the Jacobson radical of $A$ under $\varphi$ is contained in the Jacobson radical of $B$.

# Algebra Qualifying Exam - Spring 2019

1. Classify all groups of order $495$.
2. If $f(x) \in \mathbb{Q}[x]$ is irreducible, and has precisely $2$ non-real complex roots, then if $E$ is the splitting field of $f$ over $\mathbb{Q}$, show that $Gal(E/\mathbb{Q})$ is isomorphic to $S_5$.
3. If $R$ is a ring and $M$ is a Noetherian (left) $R$-module, then prove any surjective $R$-module homomorphim $\varphi : M \to M$ is an isomorphism.
4. If $R$ is an Artinian ring with no non-zero nilpotent elements, then show that $R$ is a direct sum of division rings.
5. Let $R \subset S$ be an extension of commutative rings such that $S - R$ is closed under multiplication. Prove that $R$ is integrally closed in $S$.
6. In the ring $\mathbb{C}[x, y, \frac{1}{y}]$, show that some power of $x^9 - 3y^3 + 4$ belongs to the ideal $(x^2 - y, 2x^2 - xy^2)$.

# Algebra Qualifying Exam - Fall 2019

1. Suppose $p$ and $q$ are primes with $p < q$. If $n \geq 0$ is an integer, then show that any finite group $G$ of order $pq^n$ is solvable.

2. Let $G$ be a finite group, $H \subset G$ a subgroup and $S$ a Sylow $p$-subgroup of $G$.
   (a) Show that the intersection of $H$ with some conjugate of $S$ is a Sylow $p$-subgroup of $H$.
   (b) Give an example to show that $H \cap S$ need *not* be a Sylow $p$-subgroup of $H$.

3. Give an example of a field extension of degree $4$ that has no intermediate subfield of degree 2 (hint: consider a Galois extension with group $S_4$).

4. Prove that the subset $\{(u^3, u^2v, uv^2, v^3), u, v \in \mathbb{C}\} \subset \mathbb{C}^4$ is algebraic.

5. Let $k$ be a field.
   (a) Prove that if $A, B \in M_n(k)$ are $3 \times 3$ matrices, then $A$ and $B$ are similar if and only if they have the characteristic and minimal polynomials.
   (b) Show the statement in the preceding point may fail for $4 \times 4$ matrices.

6. If $R$ is a left Noetherian ring, then show that every element $a \in A$ that admits a left inverse actually admits a 2-sided inverse.

# Algebra Qualifying Exam - January 2020

1. Classify all groups of order 75 up to isomorphism.

2. Let $G$ be a group acting transitively on a set $X$ of size $n > 1$.

   (a) If $G$ is finite, show that there exists $g \in G$ so that $gx \neq x$ for all $x \in X$ (hint: count the number of $g$ such that $gx = x$ for some $x \in X$).

   (b) Give an example to show this can fail for $G$ infinite (Hint: consider $GL_n(\mathbb{C})$ with $X$ the set of 1-dimensional subspaces of the space of column vectors).

3. Let $R$ be an integral domain with quotient field $F$.

   (a) If $M$ is a maximal ideal of $R$, show that the localization $R_M$ of $R$ at $M$ naturally embeds in $F$.

   (b) Show that $R = \cap_M R_M$ where the intersection is over all maximal ideals (hint: If $s \in \cap R_M$, let $I = \{r \in R | rs \in R\}$. show that $I$ is an ideal and is not contained in any maximal ideal $M$).

4. Let $K$ be a field of characteristic 0 containing all $m$th roots of unity. Let $L/K$ be a field extension and $a \in L$ such that $a^m \in K$. Prove that $K(a)/K$ is Galois with a Galois group that is cyclic of order dividing $m$.

5. Let $k$ be field with $f, g \in k[x_1, \ldots, x_n]$. Show that $f(a_1, \ldots, a_n) = 0$ if and only if $g(a_1, \ldots, a_n) = 0$ is equivalent to $f$ and $g$ having exactly the same (monic) irreducible factors.

6. Assume that $R$ is a semisimple ring which is a finite-dimensional algebra over a field $k$ , such that for every $r \in R$, there exists a positive integer $n = n(r)$ such that $r^n \in Z(R)$ the center of R. Prove that $R$ is commutative in the following two cases:

   (a) $k$ is finite;

   (b) $k = \mathbb{R}$. (hint: first show that there exists $x \in \mathbb{C}$ such that $x^n \notin \mathbb{R}$, for all positive $n$)

# ALGEBRA QUALIFYING EXAM FALL 2020

**PROBLEM 1.** Let $V_A$ be a $\mathbb{Q}[x]$-module that corresponds to a matrix $A \in \mathrm{Mat}_n(\mathbb{Q})$. In other words, $V_A = \mathbb{Q}^n$, and the $\mathbb{Q}[x]$-module structure is defined by $f \cdot v = f(A)v$, $f \in \mathbb{Q}[x]$, $v \in \mathbb{Q}^n$. The module $V_A$ is called cyclic if there is $v \in V$ so that $\mathbb{Q}[x] \cdot v = V_A$. The matrix $A$ is called cyclic if $V_A$ is cyclic.

(1) Prove: $V_A$ is cyclic if and only if $\mathrm{Ann}(V_A)$ is generated by the characteristic polynomial $\chi(A)$.

(2) For any matrix $A$ there is a cyclic matrix $C$ such that $AC - CA = 0$.

**PROBLEM 2.** A finite dimensional algebra over a field has finitely many up to isomorphism simple left modules.

**PROBLEM 3.** Find all pairwise non-isomorphic groups of order 147 that contain no elements of order 49.

**PROBLEM 4.** Prove that if for $f, g \in \mathbb{C}[x, y]$ the system of equations $f = g = 0$ has finitely many solutions, then the algebra $\mathbb{C}[x, y]/(f, g)$ is finite dimensional.

**PROBLEM 5.** Show that a finitely generated projective module over a principal ideal domain is free.

**PROBLEM 6.** Show that $p(x) = x^5 - 4x + 2$ is irreducible over $\mathbb{Q}$, and that it has exactly three real roots. Use this to show that the Galois group of $p(x)$ is $S_5$. Answer the question: Is $p(x) = 0$ solvable by radicals?

# ALGEBRA QUALIFYING EXAM FALL 2021

**PROBLEM 1.** Classify all groups of order $13^2 \times 7$.

**PROBLEM 2.**

(a) If $G$ is a transitive subgroup of $S_n$ with $n > 1$ which has no fixed points, prove that there exists $g \neq 1$ in $G$ so that $g$ has no fixed points.

(b). Give an example of $G < S_n$ so that every element of $G$ has fixed points but $G$ itself has no fixed points..

**PROBLEM 3.** Let $b$ be any integer coprime to 7 and consider the polynomial $f_b(x) = x^3 - 21x + 35b$. Show that $f_b$ is irreducible over $\mathbb{Q}$. Write $P$ for the set of $b \in \mathbb{Z}$ such that $b$ is coprime to 7 and the Galois group of $f_b$ is the alternating group. Find $P$. (Hint: the discriminant of the cubic polynomial of the form $x^3 + px + q$ is given by $-4p^3 - 27q^2$.)

**PROBLEM 4.** Suppose $R$ is a commutative local ring so $R$ has a unique maximal ideal $\mathfrak{m}$. Show if $x \in \mathfrak{m}$, then $1 - x$ is invertible. Show if, in addition, $R$ is Noetherian and $\mathfrak{a} \subset R$ is an ideal such that $\mathfrak{a}^2 = \mathfrak{a}$, then $\mathfrak{a} = 0$.

**PROBLEM 5.** Let $A$ be a finite dimensional central division algebra over a field $F$. Prove $[A, A] \neq A$.

**PROBLEM 6.** Let $K \subset F$ be a nontrivial finite field extension. Prove $F \otimes_K F$ is not a domain.

# ALGEBRA QUALIFYING EXAM SPRING 2022

**PROBLEM 1.** Consider the polynomial ring $\mathbb{C}[x_{ij}, 1 \le i, j \le n]$ as the algebra of polynomial functions on the space of $n \times n$ matrices $M_n(\mathbb{C})$. Let $\mathcal{N} \subset M_n(\mathbb{C})$ be the set of all nilpotent matrices. Introduce $n$ polynomials $P_j \in \mathbb{C}[x_{ij}, 1 \le i, j \le n]$, $1 \le j \le n$, defined by $P_j(A) = \mathrm{Tr} A^j$. Prove that a polynomial $Q \in \mathbb{C}[x_{ij}, 1 \le i, j \le n]$ vanishes on $\mathcal{N}$ if and only if some power of $Q$ belongs to the ideal $(P_1, P_2, ..., P_n)$.

**PROBLEM 2.** Let $R \subset S$ be an integral ring extension. Prove that $a \in R$ is invertible as an element of $R$ if and only if it is as an element of $S$.

**PROBLEM 3.** Let $R$ be a commutative ring and $M$ a finitely generated $R$-module.

(a) Prove that if $R$ is a principal ideal domain, then $M$ is projective if and only if $M$ is torsion free.

(b) Answer the question: does the assertion (a) remain valid if $R$ is assumed to be a local domain?

**PROBLEM 4.** Show that the center of a simple ring is a field and that the center of a semi-simple ring is a finite direct product of fields.

**PROBLEM 5.** Set $n = |\mathrm{SL}_2(\mathbb{F}_7)|$. For each $p|n$, find a Sylow $p$-subgroup of $\mathrm{SL}_2(\mathbb{F}_7)$.

**PROBLEM 6.** Find the Galois group of the polynomial $x^4 - 4x^2 - 21$ (over $\mathbb{Q}$). Answer the question: is this polynomial solvable in radicals?

**Algebra Exam August 2022**

Show your work. Be as clear as possible. Do all problems.

1. Let $G$ be the quaternion group of order 8.
   (a) Determine the algebra structure of $\mathbb{R}[G]$.
   (b) Determine the algebra structure of $\mathbb{C}[G]$.

2. Let $R$ be a commutative ring with 1. Let $r_1, \ldots, r_n \in R$ which generate $R$ as an ideal. Let $f : R^n \to R$ be defined by $f(a_1, \ldots, a_n) = \sum_i a_i r_i$. Show that the kernel of $f$ is a projective module.

3. Let $G$ be a finite group with a cyclic Sylow 2-subgroup $S$.
   (a) Show that $N_G(S) = C_G(S)$.
   (b) Show that if $S \neq 1$, then $G$ contains a normal subgroup of index 2 (hint: suppose that $n = [G : S]$, consider an appropriate homomorphism from $G$ to $S_n$).
   (c) Show that $G$ has a normal subgroup $N$ of odd order such that $G = NS$.

4. Let $R$ be a principal ideal domian and $p \in R$ a prime element. Suppose that $V$ is a finitely generated $R$-module with $p^a V = 0$ and suppose $v \in V$ with the annihilator of $v$ in $R$ the ideal $p^a R$. Prove that $V = Ra \oplus W$ for some submodule $W$ of $V$.

5. Let $f(x) = x^7 - 3 \in \mathbb{Q}[x]$.
   (a) Show that $f$ is irreducible in $\mathbb{Q}[x]$.
   (b) Let $K$ be the splitting field of $f$ over $\mathbb{Q}$. What is the Galois group of $K/\mathbb{Q}$.
   (c) How many subfields $L$ of $K$ are there so such that $[K : L] = 7$.

6. Let $M$ be a maximal ideal of $\mathbb{Q}[x_1, \ldots, x_t]$.
   (a) Show for each $i$, there exists a nonzero polynomial $f_i$ with coefficients in $\mathbb{Q}$ such that $f_i(x_i) \in M$.
   (b) Show that there are only finitely many maximal ideals of $\mathbb{C}[x_1, \ldots, x_t]$ which contain $M$.

**Algebra Exam January 14, 2023**

Show your work. Be as clear as possible. Do all problems.

1. Let $R = \mathbb{C}[x, y, x]/(z^2 - xy)$.
   (a) Show that $R$ is an integral domain.
   (b) Show that $R$ is integrally closed (hint: identify $R$ as an integral extension of a polynomial ring).

2. Let $G$ be a finite group and let $p$ be the smallest prime divisor of $|G|$. Assume that a Sylow $p$-subgroup $P$ of $G$ is cyclic.
   (a) Show that $N_G(P) = C_G(P)$ (hint: what is $\mathrm{Aut}(P)$?).
   (b) Show that if $G$ is solvable, then $G$ contains a subgroup $N$ of index $p$.
   (c) Show that if $N$ is a subgroup of index $p$ (whether or not $G$ is solvable), then $N$ is normal in $G$.

3. Let $F$ be a field extension of $\mathbb{Q}$ with $[F : \mathbb{Q}] = 60$ and $F/\mathbb{Q}$ Galois. Prove that if $F$ contains a 9th root of 1, then $F/\mathbb{Q}$ is a solvable.

4. Let $R$ be a finite ring with 1. Show that some element of $R$ is not the sum of nilpotent elements. Give an example to show that 1 can be a sum of nilpotent elements.

5. Let $F$ be an algebraically closed field with $A \in M_n(F)$.
   (a) Show that there exist polynomials $f(x), g(x) \in F[x]$ so that $A = f(A) + g(A)$ with $f(A)$ diagonalizable and $g(A)$ nilpotent.
   (b) Assuming (a), show that if $A = S + N$ with $S$ diagonalizable, $N$ nilpotent and $SN = NS$, then $S = f(A)$ and $N = g(A)$ (in otherwards, $S$ and $N$ are unique).

6. Let $R$ be a ring with 1. Let $M$ be a noetherian (left) $R$-module.
   (a) Show that if $f : M \to M$ is a surjective $R$-module homomorphism, then $f$ is an isomorphism.
   (b) Show that if $f : M \to M$ is an injective $R$-module homomorphism, it need not be an isomorphism.

**Algebra qualifying exam, August 2023**

*Justify all arguments completely.* Reference specific results whenever possible.

**1.** Let $R = C_{\mathrm{an}}(\mathbb{C})$ denote the ring of complex analytic functions on $\mathbb{C}$. We note that $R$ is a domain. Prove one of the following statements:

(a) The ring $R[t]$ of polynomials with coefficients in $R$ *is* a PID.

(b) The ring $R[t]$ *is not* a PID.

**2.** Let $A$ be an Artinian ring, and let $M$ be an $A$-module which is annihilated by all nilpotent ideals in $A$. Prove that $M$ is a semisimple $A$-module.

**3.** (i) Determine the radical of the ideal

$$I = \big(y^2 - 1,\ x^2 - (y+1)x + 1\big)$$

in $\mathbb{C}[x, y]$. You may write your answer in any form which allows you to easily see if a given polynomial $f(x, y)$ is in the radical or not.

(ii) Determine if the inclusion $I \subseteq \sqrt{I}$ is an equality.

**4.** Classify all groups of order 50.

**5.** Consider the polynomial $p(x) = x^{16} - \alpha x^{10} - \alpha x^6 + \alpha^2$, for $\alpha$ non-algebraic over $\mathbb{Q}$. Take $F = \mathbb{Q}(\alpha, \zeta)$, where $\zeta = e^{2\pi i/15}$.

(i) Is $p(x)$ is irreducible over $F$?

(ii) Determine the Galois group $\mathrm{Gal}(K/F)$ for the splitting field $K$ of $p(x)$ over $F$.

**6.** For a given group $G$, let $G'$ denote the subgroup generated by the commutators $[a, b] = a^{-1}b^{-1}ab$ in $G$. Explicitly,

$$G' = \langle a^{-1}b^{-1}ab : \ a, b \in G\rangle \ \subseteq \ G.$$

(i) Prove that $G'$ is normal in $G$, and that $G/G'$ is abelian.

(ii) Show that if $H$ is normal in $G$ and $G/H$ is abelian, then $H$ contains $G'$.

(iii) Prove that $S_5' = A_5$.

**Algebra qualifying exam, January 2024**

*Justify all arguments completely.* Every ring $R$ is assumed to have a unit $1 \in R$. Reference specific results whenever possible.

**1.** Let $G$ be a simple group of order 168. Show that $G$ is isomorphic to a subgroup of $A_8$, the alternating group of degree 8. Show that $G$ is not isomorphic to a subgroup of $A_6$.

**2.** Let $K$ be a field and $A$ be a finite-dimensional, semisimple $K$-algebra. Let $Z(A)$ denote the center of $A$. Prove that two finitely-generated $A$-modules $M$ and $M'$ are isomorphic as $A$-modules if and only if they are isomorphic as $Z(A)$-modules.

**3.** Let $f$ and $g$ be polynomials in $\mathbb{C}[x_1, \ldots, x_{24}]$. Suppose that for each value $z \in \mathbb{C}^{24}$ at which $f(z) = 0$, we also have $g(z) = 0$. Prove that $f$ divides some power of $g$.

**4.** Define the Jacobson radical of a ring to be the intersection of all maximal left ideals of this ring. Let $\phi : R \to S$ be a surjective morphism of rings. Prove that the image by $\phi$ of the Jacobson radical of $R$ is contained in the Jacobson radical of $S$.

**5.** Construct an example (or merely prove the existence) of a $10 \times 10$ matrix over $\mathbb{R}$ with minimal polynomial $(x+1)^2(x^4+1)$ which is not similar to a matrix over $\mathbb{Q}$.

**6.** Let $F$ be a field of characteristic not 2. Show that if $f(x) = x^8 + ax^4 + bx^2 + c$ is an irreducible polynomial over $F$ for some $a, b, c \in F$, then the Galois group of the splitting field of $f$ is solvable.