PDE Screening exam Spring 2024

December 10, 2023

Problem 1. Find a classical solution $u(x, y)$ of the equation

$$(\partial_x u)^2 - x^2 = (\partial_y u)^2 - y^2$$

in $\mathbb{R}^2$ satisfying the boundary condition $u(y, y) = y^2$.

Problem 2. Let U be open and bounded in $\mathbb{R}^n$. Suppose u and Δu are continuous in U, and u satisfies that

$$\begin{cases} \Delta u = u^4 & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

(1) If $u \geq 0$ in $\overline{U}$, prove that $u \equiv 0$.

(2) If the condition $u \geq 0$ in $\overline{U}$ is absent, what can you say about $u(x)$?

Problem 3. Let $P, Q, R, S \in \mathbb{R} \times \mathbb{R}^+$ be consecutive vertices of a rectangle with sides parallel to the lines $x + t = 0$ and $x - t = 0$, respectively.

(1) Let $u = u(x, t) \in C^2(\mathbb{R} \times \mathbb{R}^+)$ be a solution to the wave equation in $\mathbb{R} \times \mathbb{R}^+$, show that

$$u(P) + u(R) = u(Q) + u(S).$$

(2) Let $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ be given functions. Find a solution to the wave equation in $\mathbb{R}^2$ such that

$$u(t - 1, t) = \alpha(t), \quad u(5 - t, t) = \beta(t).$$