Spring 2024 Math 541b Exam

1. Let X, Y be exponential random variables with densities $f_X(t) = \lambda_1 e^{-\lambda_1 t}$, $t \geq 0$ and $f_Y(t) = \lambda_2 e^{-\lambda_2 t}$, $t \geq 0$, where $\lambda_1, \lambda_2 > 0$ are parameters. We would like to test $H_0 : \lambda_1 \leq \lambda_2$ against the alternative $H_a : \lambda_1 > \lambda_2$. Note that the null and the alternative do not change under transformation of the parameters given by $(\lambda_1, \lambda_2) \mapsto (c\lambda_1, c\lambda_2),\ c > 0$.

   (a) What is the distribution of $\frac{X}{c}$ for $c > 0$?

   (b) Suggest a function $T(x, y)$ such that
   $$T(x, y) = T(x', y') \iff x' = cx,\ y' = cy \text{ for some } c > 0,$$
   and argue that $T(X, Y)$ is a natural choice of the test statistic for the problem of testing H_0 against H_a.

   (c) Show that the density corresponding to the distribution of $T(X, Y)$ is given by $p_T(t) = \frac{\lambda_2}{\lambda_1} \frac{1}{\left(t + \frac{\lambda_2}{\lambda_1}\right)^2},\ t \geq 0$.

   (d) Note that the family of densities in part (c) depends only on one parameter $\tau = \frac{\lambda_2}{\lambda_1}$. Show that this family has the monotone likelihood ratio, and find the uniformly most powerful test for the problem $H_0' : \tau \geq 1$ against $H_a' : \tau < 1$. Which test for the original problems does this give?

2. Let $X_1, \ldots, X_n$ be an i.i.d. sample from normal distribution with mean μ and variance 1. It is known that $\mu \geq 0$.

   (a) Does the uniformly most powerful test for testing $H_0 : \mu = \mu_0$ against $H_a : \mu \neq \mu_0$ exist for any values of μ_0?

   (b) Find the simplest possible form of the Likelihood Ratio test for testing $H_0 : \mu = \mu_0$ against $H_a : \mu \neq \mu_0$ (please remember to take the fact that $\mu \geq 0$ into account!)

   (c) Let $\bar{X}_n = \frac{1}{n} \sum_{j=1}^n X_j$ be the sample mean. Assume that $\mu_0 > 0$ and prove that $\mathbb{P}\big(\bar{X}_n < 0\big) \to 0$ as $n \to \infty$ (for example, you can use Chebyshev's inequality). Use this fact to show directly that the asymptotic distribution of $2 \log \Lambda_n$, where $\Lambda_n = \frac{\sup_{\mu \geq 0} L_n(\mu)}{L_n(\mu_0)}$ and $L_n(\mu)$ is the likelihood function, is chi-squared with 1 degree of freedom (again, assuming that $\mu_0 > 0$).

   (d) Find the asymptotic distribution of $2 \log \Lambda_n$ when $\mu_0 = 0$.