Spring 2024 Math 541a Exam

Problem 1. Suppose that $X_1, \dots, X_n$ are independent and identically distributed (i.i.d.) random variables with the common distribution $N(\theta, \theta^2)$, where $\theta \in (0, \infty)$ is the unknown parameter. Let $W_n = n^{-1} \sum_{i=1}^n X_i^2$.

Fact: if $X \sim N(\mu, \sigma^2)$, then $\mathbb{E}X^3 = \mu^3 + 3\mu\sigma^2$ and $\mathbb{E}X^4 = 3\sigma^4 + 6\sigma^2\mu^2 + \mu^4$.

1. Find a sequence of constants a_n and a function $b(\theta)$ such that $a_n(W_n - b(\theta)) \xrightarrow{d} T$, where T is a non-degenerate random variable with mean zero. What is the variance of T? Here, a non-degenerate random variable means its variance is strictly positive.

2. Find the MLE $\hat{\theta}$ of θ.

3. Find a sequence of constants c_n and a function $d(\theta)$ such that $c_n(\hat{\theta} - d(\theta)) \xrightarrow{d} U$, where U is a non-degenerate random variable with mean zero. What is the variance of U? [Hint: consider the multivariate Δ-method.]

4. Find a non-degenerate variance stabilizing transformation h of W_n, i.e., the asymptotic non-zero variance of $h(W_n)$ does not depend on θ.

Problem 2. The Information Inequality. Let $\boldsymbol{X}$ be an observation in $\mathbb{R}^n$ with density $p(\boldsymbol{x}, \theta)$ for $\theta \in \Theta \subset \mathbb{R}^p$, and $T = T(\boldsymbol{X})$ an estimator.

1. Use the Cauchy Schwarz inequality to prove the information inequality

$$\text{Var}(T) \geq \frac{[\dot{g}(\theta)]^2}{I(\theta)}$$

when $p = 1$, that is, when the parameter space is a subset of the real line, and where $g(\theta) = E_\theta[T]$ and $\dot{g}$ is its derivative with respect to θ, and $I(\theta) = \text{Var}_\theta(U(\theta, \boldsymbol{X})$ where

$$U(\theta, \boldsymbol{x}) = \frac{\partial}{\partial \theta} \log p(x, \theta).$$

2. Consider now the Information Inequality when $\theta \in \mathbb{R}^d$ and $T \in \mathbb{R}^{1 \times r}$ is an estimate of r functions of θ. Under regularity on the density, we obtain

$$\text{Var}_\theta(T(\boldsymbol{X})^\top) \geq \dot{g}(\theta)^\top \mathbf{I}(\theta)^{-1} \dot{g}(\theta),$$

where each column of $\dot{g}$ is the gradient with respect to θ of the row entry of g, and where the information matrix is now the 'variance-covariance' matrix of the score function.

Let $\boldsymbol{X}$ be a multivariate normal observation with unknown mean μ and known invertible covariance matrix Σ, that is, with density

$$p(x, \mu) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\right)$$

Compute the information lower bound for the unbiased estimation of

$$g(\mu) = \frac{1}{2} \sum_{i=1} \mu_i^2$$