Geometry and Topology Screening Exam
Spring 2024

Instructions: Solve as many of the following 7 problems as you can. Solutions will be graded for correctness, completeness and clarity. Partial credit will be awarded for partial solutions indicating clear progress.

Problem 1. Let M be a path-connected smooth manifold. Prove that for any $p_1, p_2 \in M$, there is a diffeomorphism φ of M with $\varphi(p_1) = p_2$.

Problem 2. Show that for any vector bundle $p\colon E \to B$ over a space B, the map p is a homotopy equivalence.

Problem 3. Let X be a closed manifold, and let θ be a degree 1 de Rham class such that

$$\int_\Gamma \theta \in \mathbb{Z} \qquad \text{for any cycle } \Gamma.$$

Show that there exists a map $\varphi\colon X \to S^1$ so that the induced map

$$\varphi^*\colon \mathrm{H}^1_{\mathrm{dR}}(S^1) \to \mathrm{H}^1_{\mathrm{dR}}(X)$$

contains θ in its image.

Problem 4. Suppose $f\colon M \to N$ is a smooth map between connected smooth manifolds which is both an immersion and a submersion. Is f necessarily a diffeomorphism? Prove it or provide a counter-example.

Problem 5. Let 0_n be the $n \times n$ 0-matrix and let I_n be the $n \times n$ identity matrix. Let J be the $2n \times 2n$ matrix given by

$$J = \begin{bmatrix} 0_n & -I_n \\ I_n & 0_n \end{bmatrix}$$

Finally, let $G \subset \mathrm{GL}_{2n}(\mathbb{R})$ be the group of invertible $2n \times 2n$ matrices A that commute with J. Show that G is a sub-manifold of $\mathrm{GL}_{2n}(\mathbb{R})$ and compute its dimension.

Problem 6. Let X be a closed smooth manifold and let V be a vector-field. Suppose that α and β are closed 1-forms on X such that $\alpha(V)$ and $\beta(V)$ are constant functions. Show that

$$\alpha \wedge \beta$$

is invariant under the flow generated by V.

Problem 7. Let $K \subset S^3$ be a smoothly embedded S^1. Compute the first homology group of $S^3 \setminus K$ over $\mathbb{Z}$.