Geometry/Topology Qualifying Exam

Spring 2012

Solve all **SEVEN** *problems. Partial credit will be given to partial solutions.*

1. (10 pts) Prove that a compact smooth manifold of dimension n cannot be immersed in $\mathbb{R}^n$.

2. (10 pts) Let $\Sigma_{1,1}$ be the compact oriented surface with boundary, obtained from $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with coordinates (x, y) by removing a small disk $\{(x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 = \frac{1}{100}\}$.
   (a) Compute the homology of $\Sigma_{1,1}$.
   (b) Let Σ_2 denote a closed oriented surface of genus 2. Use your answer from (a) to compute the homology of Σ_2.

3. (10 pts) Let S be an oriented embedded surface in $\mathbb{R}^3$ and ω be an area form on S which satisfies $\omega(p)(e_1, e_2) = 1$ for all $p \in S$ and any orthonormal basis (e_1, e_2) of T_pS with respect to the standard Euclidean metric on $\mathbb{R}^3$. If (n_1, n_2, n_3) is the unit normal vector field of S, then prove that
$$\omega = n_1 dy \wedge dz - n_2 dx \wedge dz + n_3 dx \wedge dy,$$
   where (x, y, z) are the standard Euclidean coordinates on $\mathbb{R}^3$.

4. (10 pts) Consider the space $X = M_1 \cup M_2$, where M_1 and M_2 are Möbius bands and $M_1 \cap M_2 = \partial M_1 = \partial M_2$. Here a *Möbius band* is the quotient space $([-1,1]\times[-1,1])/((1,y) \sim (-1,-y))$.
   (a) Determine the fundamental group of X.
   (b) Is X homotopy equivalent to a compact orientable surface of genus g for some g?

5. (10 pts) Determine all the connected covering spaces of $\mathbb{RP}^{14} \vee \mathbb{RP}^{15}$.

6. (10 pts) Let $f : M \to N$ be a smooth map between smooth manifolds, X and Y be smooth vector fields on M and N, respectively, and suppose that $f_*X = Y$ (i.e., $f_*(X(x)) = Y(f(x))$ for all $x \in M$). Then prove that $f^*(\mathcal{L}_Y\omega) = \mathcal{L}_X(f^*\omega)$, where ω is a 1-form on N. Here $\mathcal{L}$ denotes the Lie derivative.

7. (10 pts) Consider the linearly independent vector fields on $\mathbb{R}^4 - \{0\}$ given by:
$$X(x_1, x_2, x_3, x_4) = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial x_4}$$
$$Y(x_1, x_2, x_3, x_4) = -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} - x_4\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_4}.$$
   Is the rank 2 distribution orthogonal to these two vector fields integrable? Here orthogonality is measured with respect to the standard Euclidean metric on $\mathbb{R}^4$.

Geometry-Topology Qualitfying exam
Fall 2012

Solve all of the problems. Partial credit will be given for partial answers.

1. Denote by $S^1 \subset \mathbf{R}^2$ the unit circle and consider the torus $T^2 = S^1 \times S^1$. Now, define $A \subset T^2 = S^1 \times S^1$ by

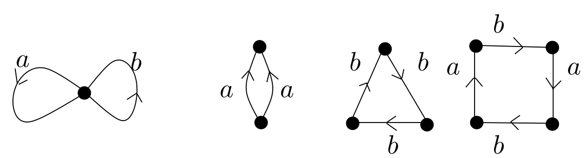
$$A = \{(x, y, z, w) \in T^2 \mid (x, y) = (0, 1) \text{ or } (z, w) = (0, 1)\ \}.$$

Compute $H^*(T^2, A)$. *Here we regard S^1 as a subset of the plane, hence we indicate points on S^1 as ordered pairs.*

2. Denote by S^1 and S^2 the circle and sphere respectively. Recall that the definition of the smash product $X \wedge Y$ of two pointed spaces is the quotient of $X \times Y$ by $(x, y_0) \sim (x_0, y)$.

Show that $S^1 \times S^1$ and $S^1 \wedge S^1 \wedge S^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not.

3. Let X be a CW-complex with one vertex, two one cells and 3 two cells whose attaching maps are indicated below.



1-skeleton   2-skeleton

   (a) Compute the homology of X.
   (b) Present the fundamental group of X and prove its nonabelian.

(*Justify your work.*)

4. Does there exist a smooth embedding of the projective plane $\mathbf{R}P^2$ into $\mathbf{R}^2$? Justify your answer.

5. Let M be a manifold, and let $C^\infty(M)$ be the algebra of C^∞ functions $M \to \mathbf{R}$. Explain the relationship between vector fields on M and $C^\infty(M)$. If we consider the vector fields X and Y as maps $C^\infty(M) \to C^\infty(M)$ is the composition map XY also a vector field? What about $[X, Y] = XY - YX$? Explain.

6. Let S be the unit sphere defined by $x^2 + y^2 + z^2 + w^2 = 1$ in $\mathbf{R}^4$. Compute $\int_S \omega$ where $\omega = (w + w^2) dx \wedge dy \wedge dz$.

7. Does the equation $x^2 = y^3$ define a smooth submanifold in $\mathbf{R}^3$? Prove your claim.

GEOMETRY TOPOLOGY QUALITFYING EXAM
SPRING 2013

Solve all of the problems that you can. Partial credit will be given for partial solutions.

(1) Consider the form
$$\omega = (x^2 + x + y)dy \wedge dz$$
on $\mathbb{R}^3$. Let $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ be the unit sphere, and $i \colon S^2 \to \mathbb{R}^3$ the inclusion.
   (a) Calculate $\int_{S^2} \omega$.
   (b) Construct a closed form α on $\mathbb{R}^3$ such that $i^*\alpha = i^*\omega$, or show that such a form α does not exist.

(2) Find all points in $\mathbb{R}^3$ in a neighborhood in which the functions $x, x^2 + y^2 + z^2 - 1, z$ can serve as a local coordinate system.

(3) Prove that the real projective space $\mathbb{R}P^n$ is a smooth manifold of dimension n.

(4) (a) Show that every closed 1-form on S^n, $n > 1$ is exact.
   (b) Use this to show that every closed 1-form on $\mathbb{R}P^n$, $n > 1$ is exact.

(5) Let X be the space obtained from $\mathbb{R}^3$ by removing the three coordinate axes. Calculate $\pi_1(X)$ and $H_*(X)$.

(6) Let $X = T^2 - \{p, q\}$, $p \neq q$ be the twice punctured 2-dimensional torus.
   (a) Compute the homology groups $H_*(X, \mathbb{Z})$.
   (b) Compute the fundamental group of X.

(7) (a) Find all of the 2-sheeted covering spaces of $S^1 \times S^1$.
   (b) Show that if a path-connected, locally path connected space X has $\pi_1(X)$ finite, then every map $X \to S^1$ is nullhomotopic.

(8) (a) Show that if $f \colon S^n \to S^n$ has no fixed points then $\deg(f) = (-1)^{n+1}$.
   (b) Show that if X has S^{2n} as universal covering space then $\pi_1(X) = \{1\}$ or $\mathbb{Z}_2$.

Date: February 1, 2013.

Geometry/Topology Qualifying Exam

Fall 2013

Solve all **SEVEN** *problems. Partial credit will be given to partial solutions.*

1. (15 pts) Let X denote S^2 with the north and south poles identified.
   (a) (5 pts) Describe a cell decomposition of X and use it to compute $H_i(X)$ for all $i \geq 0$.
   (b) (5 pts) Compute $\pi_1(X)$.
   (c) (5 pts) Describe (i.e., draw a picture of) the universal cover of X and all other connected covering spaces of X.

2. (10 pts) Show that if M is compact and N is connected, then every submersion $f : M \to N$ is surjective.

3. (10 pts) Show that the orthogonal group $O(n) = \{A \in M_n(\mathbb{R}) \mid AA^T = id\}$ is a smooth manifold. Here $M_n(\mathbb{R})$ is the set of $n \times n$ real matrices.

4. (10 pts) Compute the de Rham cohomology of $S^1 = \mathbb{R}/\mathbb{Z}$ from the definition.

5. (10 pts) Let X, Y be topological spaces and $f, g : X \to Y$ two continuous maps. Consider the space Z obtained from the disjoint union $(X \times [0,1]) \sqcup Y$ by identifying $(x, 0) \sim f(x)$ and $(x, 1) \sim g(x)$ for all $x \in X$. Show that there is a long exact sequence of the form:
$$\cdots \to H_n(X) \to H_n(Y) \to H_n(Z) \to H_{n-1}(X) \to \ldots$$

6. (10 pts) A lens space $L(p, q)$ is the quotient of $S^3 \subset \mathbb{C}^2$ by the $\mathbb{Z}/p\mathbb{Z}$-action generated by $(z_1, z_2) \mapsto (e^{2\pi i/p} z_1, e^{2\pi iq/p} z_2)$ for coprime p, q.
   (a) (5 pts) Compute $\pi_1(L(p, q))$.
   (b) (5 pts) Show that any continuous map $L(p, q) \to T^2$ is null-homotopic.

7. (10 pts) Consider the space of all straight lines in $\mathbb{R}^2$ (not necessarily those passing through the origin). Explain how to give it the structure of a smooth manifold. Is it orientable?

Geometry and Topology Graduate Exam
Spring 2014

Solve all **SEVEN** *problems. Partial credit will be given to partial solutions.*

Problem 1. Let X_n denotes the complement of n distinct points in the plane $\mathbb{R}^2$. Does there exist a covering map $X_2 \to X_1$? Explain.

Problem 2. Let $D = \{z \in \mathbb{C}; |z| \leq 1\}$ denote the unit disk, and choose a base point z_0 in the boundary $S^1 = \partial D = \{z \in \mathbb{C}; |z| = 1\}$. Let X be the space obtained from the union of D and $S^1 \times S^1$ by gluing each $z \in S^1 \subset D$ to the point $(z, z_0) \in S^1 \times S^1$. Compute all homology groups $H_k(X; \mathbb{Z})$.

Problem 3. Let $B^n = \{x \in \mathbb{R}^n; \|x\| \leq 1\}$ denote the n–dimensional closed unit ball, with boundary $S^{n-1} = \{x \in \mathbb{R}^n; \|x\| = 1\}$. Let $f\colon B^n \to \mathbb{R}^n$ be a continuous map such that $f(x) = x$ for every $x \in S^{n-1}$. Show that the origin 0 is contained in the image $f(B^n)$. (Hint: otherwise, consider $S^{n-1} \to B^n \xrightarrow{f} \mathbb{R}^n - \{0\}$.)

Problem 4. Consider the following vector fields defined in $\mathbb{R}^2$:

$$\mathbf{X} = 2\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \qquad \text{and} \qquad \mathbf{Y} = \frac{\partial}{\partial y}.$$

Determine whether or not there exists a (locally defined) coordinate system (s, t) in a neighborhood of $(x, y) = (0, 1)$ such that

$$\mathbf{X} = \frac{\partial}{\partial s}, \qquad \text{and} \qquad \mathbf{Y} = \frac{\partial}{\partial t}.$$

Problem 5. Let M be a differentiable (not necessarily orientable) manifold. Show that its cotangent bundle

$$T^*M = \{(x, u); x \in M \text{ and } u\colon T_xM \to \mathbb{R} \text{ linear}\}$$

is a manifold, and is orientable.

Problem 6. Calculate the integral $\int_{S^2} \omega$ where S^2 is the standard unit sphere in $\mathbb{R}^3$ and where ω is the restriction of the differential 2–form

$$(x^2 + y^2 + z^2)(x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy)$$

Problem 7. Let M be a compact m–dimensional submanifold of $\mathbb{R}^m \times \mathbb{R}^n$. Show that the space of points $x \in \mathbb{R}^m$ such that $M \cap \mathbb{R}^n$ is infinite has measure 0 in $\mathbb{R}^m$.

Geometry/Topology Qualifying Exam - Fall 2014

1. Show that if (X, x) is a pointed topological space whose universal cover exists and is compact, then the fundamental group $\pi_1(X, x)$ is a finite group.

2. Recall that if (X, x) and (Y, y) are pointed topological spaces, then the wedge sum (or 1-point union) $X \vee Y$ is the space obtained from the disjoint union of X and Y by identifying x and y. Show that T^2 (the 2-torus $S^1 \times S^1$) and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups, but are not homeomorphic.

3. Suppose S^n is the standard unit sphere in Euclidean space and that $f : S^n \to S^n$ is a continuous map.

   i) Show that if f has no fixed points, then f is homotopic to the antipodal map.

   ii) Show that if $n = 2m$, then there exists a point $x \in S^{2m}$ such that either $f(x) = x$ or $f(x) = -x$.

4. If M is a smooth manifold of dimension d, using basic properties of de Rham cohomology, describe the de Rham cohomology groups $H^*_{dR}(S^1 \times M)$ in terms of the groups $H^*_{dR}(M)$ (along the way, please explain, quickly and briefly, how to compute $H^*_{dR}(S^1)$).

5. Show that if $X \subset \mathbb{R}^3$ is a closed (i.e., compact and without boundary) submanifold that is homeomorphic to a sphere with $g > 1$ handles attached, then there is a non-empty open subset on which the Gaussian curvature K is negative.

6. Suppose M is a (non-empty) closed oriented manifold of dimension d. Show that if ω is a differential d-form, and X is a (smooth) vector field on X, then the differential form $\mathfrak{L}_X \omega$ necessarily vanishes at some point of M.

7. Let V be a 2-dimensional complex vector space, and write $\mathbb{CP}^1$ for the set of complex 1-dimensional subspaces of V. By explicit construction of an atlas, show that $\mathbb{CP}^1$ can be equipped with the structure of an oriented manifold.

Geometry and Topology Graduate Exam
Fall 2015

Problem 1. (15 points)

(a) Define the two notions "homotopy between two maps" and "homotopy equivalences between two spaces".

(b) Give an example of two topological spaces X and Y that are homotopy equivalent but are not homeomorphic.

(c) Give an example of path-connected topological spaces X and Y that have isomorphic fundamental groups but are not homotopy equivalent.

(d) Give an example of path-connected topological spaces X and Y that have isomorphic first homology groups $H_1(X;\mathbb{Z}) \cong H_1(Y;\mathbb{Z})$ but whose fundamental groups are not isomorphic.

Problem 2. (15 points) Let T be the 2–dimensional torus, and let K be the Klein bottle.

(a) Describe a twofold covering map $p\colon T \to K$. ("Twofold" means that the preimage of each point of K consists of two points of T.)

(b) Pick base points $x_0 \in T$ and $y_0 \in K$ such that $y_0 = p(x_0)$. Give generators for the fundamental groups $\pi_1(T;x_0)$ and $\pi_1(K;y_0)$ and, for each generator of $\pi_1(T;x_0)$, express its image under the induced homomorphism $p_*\colon \pi_1(T;x_0) \to \pi_1(K;y_0)$ in terms of the generators of $\pi_1(K;y_0)$.

Problem 3. (25 points) Let Σ_g and $\Sigma_{g'}$ be closed orientable surfaces of genus g and $g' > 0$, respectively. Let $f\colon B^2 \to \Sigma_g$ be an embedding of the 2–dimensional disk B^2, and consider the simple closed curve $\gamma = f(S^1) \subset \Sigma_g$. Similarly, let $\gamma' = f'(S^1) \subset \Sigma_{g'}$ be associated to an embedding $f'\colon B^2 \to \Sigma_{g'}$. Finally, let X be the topological space obtained by gluing Σ_g and $\Sigma_{g'}$ along γ and γ'; namely, X is obtained from the disjoint union $\Sigma_g \sqcup \Sigma_{g'}$ by gluing $f(x)$ to $f'(x)$ for every $x \in S^1$.

(a) Compute the fundamental group of X.

(b) Compute all homology groups of X.

(c) Is X homotopy equivalent to the product $\Sigma_g \times \Sigma_{g'}$?

Problem 4. (15 points) Let M be a manifold of dimension n, and let ω be a differential form of degree $n-1$ on M. Suppose that $\int_N \omega = 0$ for every $(n-1)$–dimensional oriented closed submanifold N of M. Show that $dw = 0$. (Possible hint: look at small spheres.)

Problem 5. (15 points) Consider the vector fields $\mathbf{v} = \partial_x + xz\partial_z$ and $\mathbf{w} = \partial_y + yz\partial_z$ in $\mathbb{R}^3$. If P is a point of $\mathbb{R}^3$, does there exist a local coordinate system in a neighborhood of P in which $\mathbf{v}$ and $\mathbf{w}$? Namely, is there a diffeomorphism $\phi\colon U \to V$ from a neighborhood U of P to an open subset $V \subset \mathbb{R}^3$ that sends $\mathbf{v}$ to ∂_x and $\mathbf{w}$ to ∂_y?

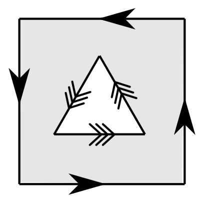
Problem 6. (15 points) Let $f\colon \mathbb{C} \to \mathbb{C}$ be a polynomial of one complex variable. Recall that the one-point compactification $\mathbb{C} \cup \{\infty\}$ of $\mathbb{C}$ is homeomorphic to the sphere S^2.

(a) Show that f extends to a continuous map $\bar{f}\colon S^2 \to S^2$.

(b) Show that the degree of $\bar{f}$ (in the sense of topology or geometry) is equal to the degree of the polynomial f (in the algebraic sense).

Geometry and Topology Graduate Exam
Spring 2016

Problem 1. Let Y be the space obtained by removing an open triangle from the interior of a compact square in $\mathbb{R}^2$. Let X be the quotient space of Y by the equivalence relation which identifies all four edges of the square and which identifies all three edges of the triangle according to the diagram below. Compute the fundamental group of X.



Problem 2. Let X be a path connected space with $\pi_1(X; x_0) = \mathbb{Z}/5$, and consider a covering space $\pi : \widetilde{X} \to X$ such that $p^{-1}(x_0)$ consists of 6 points. Show that $\widetilde{X}$ has either 2 or 6 connected components.

Problem 3. Compute the homology groups $H_k(S^1 \times S^n; \mathbb{Z})$ of the product of the circle S^1 and the sphere S^n, with $n \geq 1$.

Problem 4. Let M be a compact oriented manifold of dimension n, and consider a differentiable map $f\colon M \to \mathbb{R}^n$ whose image $f(M)$ has non-empty interior in $\mathbb{R}^n$.

(a) Show there there is at least one point $x \in M$ where f is a local diffeomorphism, namely such that there exists an open neighborhood $U \subset M$ of x such that restriction $f_{|U}\colon U \to f(U)$ is a diffeomorphism.

(b) Show that there exists at least two points x, $y \in M$ such that f is a local diffeomorphism at x and y, f is orientation-preserving at x, and f is orientation-reversing at y. Possible hint: What is the degree of f?

Problem 5. Consider the real projective space $\mathbb{RP}^n$, quotient of the sphere S^n by the equivalence relation that identifies each $x \in S^n$ to $-x$. Is there a degree n differential form such $\omega \in \Omega^n(\mathbb{RP}^n)$ such that $\omega(y) \neq 0$ at every $y \in \mathbb{RP}^n$? (The answer may depend on n.)

Problem 6. Let S^n denote the n–dimensional sphere, and remember that for $n \geq 1$ its de Rham cohomology groups are

$$H^k(S^n) \cong \begin{cases} 0 & \text{if } k \neq 0, n \\ \mathbb{R} & \text{if } k = 0, n. \end{cases}$$

Consider a differentiable map $f : S^{2n-1} \to S^n$, with $n \geq 2$. If $\alpha \in \Omega^n(S^n)$ is a differential form of degree n on S^n such that $\int_{S^n} \alpha = 1$, let $f^*(\alpha) \in \Omega^n(S^{2n-1})$ be its pull-back under the map f.

(a) Show that there exists $\beta \in \Omega^{n-1}(S^{2n-1})$ such that $f^*(\alpha) = d\beta$.

(b) Show that the integral $I(f) = \int_{S^{2n-1}} \beta \wedge d\beta$ is independent of the choice of β and α.

Geometry and Topology Graduate Exam
Spring 2017

Problem 1. Let $\omega \in \Omega^2(M)$ be a differential form of degree 2 on a $2n$–dimensional manifold M. Suppose that ω is exact, namely that $\omega = d\alpha$ for some $\alpha \in \Omega^1(M)$. Show that $\omega^n = \omega \wedge \omega \wedge \cdots \wedge \omega \in \Omega^{2n}(M)$ is exact.

Problem 2. Consider the unit disk $B^2 = \{x \in \mathbb{R}^2; \|x\| \leq 1\}$ and the circle $S^1 = \{x \in \mathbb{R}^2; \|x\| = 1\}$. The two manifolds $U = S^1 \times B^2$ and $V = B^2 \times S^1$ have the same boundary $\partial U = \partial V = S^1 \times S^1$. Let X be the space obtained by gluing U and V along this common boundary; namely, X is the quotient of the disjoint union $U \sqcup V$ under the equivalence relation that identifies each point of ∂U to the point of ∂V that corresponds to the same point of $S^1 \times S^1$.

Compute the fundamental group $\pi_1(X; x_0)$.

Problem 3. Compute the homology groups $H_n(X; \mathbb{Z})$ of the topological space X of Problem 2.

Problem 4. For a unit vector $v \in S^{n-1} \subset \mathbb{R}^n$, let $\pi_v \colon \mathbb{R}^n \to v^\perp$ be the orthogonal projection to its orthogonal hyperplane $v^\perp \subset \mathbb{R}^n$. Let M be an m–dimensional submanifold of $\mathbb{R}^n$ with $m \leq \frac{n}{2} - 1$. Show that, for almost every $v \in S^{n-1}$, the restriction of π_v to M is injective.

Possible hint: Use a suitable map $f \colon M \times M - \Delta \to S^{n-1}$, where $\Delta = \{(x, x); x \in M\}$ is the diagonal of $M \times M$.

Problem 5. Let G be a topological group. Namely, G is simultaneously a group and a topological space, the multiplication map $G \times G \to G$ defined by $(g, h) \mapsto gh$ is continuous, and the inverse map $G \to G$ defined by $g \mapsto g^{-1}$ is continuous as well. Show that, if $e \in G$ is the identity element of G, the fundamental group $\pi_1(G; e)$ is abelian.

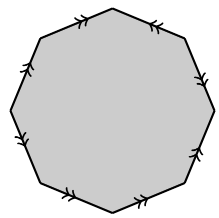
Problem 6. Let $f \colon M \to N$ be a differentiable map between two compact connected oriented manifolds M and N of the same dimension m. Show that, if the induced homomorphism $H_m(f) \colon H_m(M; \mathbb{Z}) \to H_m(N; \mathbb{Z})$ is nonzero, the subgroup $f_*\big(\pi_1(M; x_0)\big)$ has finite index in $\pi_1(N; f(x_0))$. Hint: consider a suitable covering of N.

Geometry and Topology Graduate Exam
Fall 2017

Solve all **SEVEN** *problems. Partial credit will be given to partial solutions.*

Problem 1. Let M be an oriented compact m–dimensional manifold, and let $f\colon M \to \mathbb{R}^m$ be a smooth map. Show that, for almost every $y \in \mathbb{R}^m$ (meaning, for y in the complement of a set of measure 0), the preimage $f^{-1}(y)$ consists of an *even* number of points.

Problem 2.
The sides of an octagon are glued using the pattern below. Determine the fundamental group of the associated quotient space.



Problem 3. Let $p\colon \widetilde{X} \to X$ be a covering map with X path connected and locally path connected, and with $\pi_1(X; x_0) \cong \mathbb{Z}/5$. Show that, if the fiber $p^{-1}(x_0)$ consists of 4 points, the covering is trivial.

Problem 4. Consider the following two-dimensional distribution on $\mathbb{R}^3$:

$$\mathcal{D} = \ker(2dx - e^y dz).$$

Is there a neighborhood U of $0 \in \mathbb{R}^3$, along with a coordinate system (w, s, t) on U, such that $\mathcal{D}|_U = \mathrm{span}(\frac{\partial}{\partial w}, \frac{\partial}{\partial s})$? Justify your answer (with a proof).

Problem 5. Show that the subset

$$M = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4; x_1^2 + x_2^2 = x_3^2 + x_4^2\}$$

is *not* a differentiable submanifold of $\mathbb{R}^4$.

Problem 6. Let X be the subspace of $\mathbb{R}^3$ defined by

$$X = \left\{(x, y, z) \in \mathbb{R}^3 : (x^2 + y^2 - 1)(x^2 + z^2 - (\tfrac{1}{2})^2) = 0, \right\}$$

so that X is the union of two cylinders of radius 1 along the z-axis and a cylinder of radius $\frac{1}{2}$ along the y-axis. Determine the homology groups $H_*(X)$.

Problem 7. Consider the 2-form on $\mathbb{R}^3 \backslash \{(0, 0, 0)\}$ given by

$$\sigma = \frac{1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \left(x\, dy \wedge dz - y\, dx \wedge dz + z\, dx \wedge dy\right).$$

Show that σ is closed but not exact. (Possible hint: Integrate σ over the sphere S^2).

Geometry and Topology Graduate Exam
Spring 2018

Solve all **7** *problems. Partial credit will be given to partial solutions.*

Problem 1.

a) Define $F\colon \mathbb{R}^4 \to \mathbb{R}^2$ by
$$F(x_1, x_2, x_3, x_4) = (x_1^2 + x_2^2 + x_3^2 + x_4^2, x_1^2 + x_2^2 - x_3^2 - x_4^2).$$
Show that $M = F^{-1}(1, 0)$ is a smooth manifold.

b) For each $x = (x_1, x_2, x_3, x_4) \in M$ show that the tangent space T_xM is spanned by $(x_2, -x_1, 0, 0)$ and $(0, 0, x_4, -x_3)$.

c) Let $G\colon \mathbb{R}^4 \to \mathbb{R}$ be a smooth map and $g = G\mid_M$ be the restriction of G to M. Show that x is a critical point of g if and only if $\ker dF_x \subset \ker dG_x$.

d) If $G(x_1, x_2, x_3, x_4) = x_1 + x_3$ find the critical points of g.

Problem 2. Let X be a topological space. Let SX denote the suspsension of X, i.e. the space obtained from $X \times [0, 1]$ by collapsing $X \times \{0\}$ to a point and $X \times \{1\}$ to another point:
$$SX := X\times[0, 1]/\sim, \quad \text{where } \{(x, t) \sim (y, s) \text{ if } s = t = 0 \text{ or } s = t = 1 \text{ or } (x, t) = (y, s)\}$$
Determine the relationship between the homology of SX and X.

Problem 3. Consider $\mathbb{R}^3$ with the coordinates (x, y, z). Write down explicit formulas for the vector fields X and Y which represent the infinitesimal generators of rotation about the x and y axes respectively and compute their Lie bracket.

Problem 4.

Draw a based covering space of the figure eight with each of the following subgroups of $\mathbb{Z} * \mathbb{Z} = \langle a, b\rangle$ as its fundamental group. In each case determine whether or not the subgroup is normal.

(1) $\langle a^3, b, aba^{-1}, a^{-1}ba\rangle$
(2) $\langle a^2, b^2, aba, bab\rangle$

Problem 5.

a) Prove there does not exist a degree 1 map $S^2 \to T^2$.

b) Let $f : M^p \to N^p$ be any smooth map between two connected compact orientable manifolds of the same dimension p. Suppose there exists a regular value $y \in N$ with three pre-images $f^{-1}(y) = \{x_1, x_2, x_3\}$. Prove that f is necessarily surjective.

Problem 6.

Let $T = S^1 \times S^1$ and let $x, y \in T$ be two distinct points. Let Y be the quotient space obtained from $T \times \{1, 2\}$ by identifying the points $(x, 1)$ and $(x, 2)$ into a single point $\bar{x}$, and identifying the points $(y, 1)$ and $(y, 2)$ into a single point $\bar{y}$, distinct from $\bar{x}$. Compute $\pi_1(Y, \bar{x})$.

ADDITIONAL PROBLEM ON NEXT PAGE

Problem 7.

a) Compute the integral

$$\int_{S^2} 2xyz dx \wedge dy + (yz + xy^2) dx \wedge dz + xz dy \wedge dz,$$

where S^2 is oriented as the boundary of the unit ball $B^3 \subset \mathbb{R}^3$.

b) Show that the form

$$\omega = \frac{x dy - y dx}{x^2 + y^2}$$

on $\mathbb{R}^2 - \{(0,0)\}$ is closed but not exact.

Geometry and Topology Graduate Exam
Fall 2018

Solve all **6** problems. Partial credit will be given to partial solutions.

Problem 1. Let $p : \tilde{X} \to X$ denote the universal cover of the space X, and let $Y \subset X$ be a path-connected subspace. Suppose that the preimage $p^{-1}(Y)$ is path connected. Show that, for an arbitrary base point $y_0 \in Y$, the homomorphism $i_* : \pi_1(Y; y_0) \to \pi_1(X; y_0)$ induced by the inclusion map $i\colon Y \to X$ is surjective.

Problem 2. Consider the Klein bottle

$$K = S^1 \times [0,1]/\sim$$

where, if S^1 is the unit circle in the complex plane $\mathbb{C}$ and if $\bar{z}$ denotes the complex conjugate of $z \in S^1$, the equivalence relation $\sim$ identifies each $(z,1) \in S^1 \times \{1\}$ to $(\bar{z}, 0) \in S^1 \times \{0\}$. Compute all homology groups $H_n(K; \mathbb{Z}_4)$ with coefficients in the cyclic group $\mathbb{Z}_4$ of order 4.

Problem 3. Consider the $2n \times 2n$ matrix

$$J = \begin{pmatrix} 0_n & \mathrm{I}_n \\ -\mathrm{I}_n & 0_n \end{pmatrix}$$

where I_n denotes the identity matrix of order n, and where 0_n represents the square matrix of order n whose entries are all equal to 0. In the vector space $\mathrm{M}_{2n}(\mathbb{R}) \cong \mathbb{R}^{4n^2}$ of $2n \times 2n$ matrices, set

$$\Sigma_n = \{A \in \mathrm{M}_{2n}(\mathbb{R}); AJA^{\mathrm{t}} = J\} = f^{-1}(J)$$

for the map $f\colon \mathrm{M}_{2n}(\mathbb{R}) \to \mathrm{M}_{2n}(\mathbb{R})$ defined by $f(A) = AJA^{\mathrm{t}}$.

- **a.** Let $T_{\mathrm{I}_{2n}} f\colon \mathrm{M}_{2n}(\mathbb{R}) \to \mathrm{M}_{2n}(\mathbb{R})$ denote the tangent map (= differential map) of f at the identity matrix $\mathrm{I}_{2n} \in \mathrm{M}_{2n}(\mathbb{R})$ where, since $\mathrm{M}_{2n}(\mathbb{R})$ is a vector space, we use the canonical identification between the tangent space $T_{\mathrm{I}_{2n}}\mathrm{M}_{2n}(\mathbb{R})$ and $\mathrm{M}_{2n}(\mathbb{R})$. Determine the dimension of the image of $T_{\mathrm{I}_{2n}} f$.
- **b.** Show that there is a neighborhood U of I_{2n} in $\mathrm{M}_{2n}(\mathbb{R})$ such that $\Sigma_n \cap U$ is a submanifold of $\mathrm{M}_{2n}(\mathbb{R}) \cong \mathbb{R}^{4n^2}$. What is its dimension?

Problem 4. Let M be an m–dimensional submanifold of $\mathbb{R}^n$. Show that, if $n > 2m + 1$, there exists a hyperplane $H \subset \mathbb{R}^n$ such that the restriction of the orthogonal projection $\pi_H\colon \mathbb{R}^n \to H$ to M is injective. Possible hint: consider the map which associates the vector $\overrightarrow{PQ}/\|\overrightarrow{PQ}\|$ to each pair $(P,Q) \in M \times M$ with $P \neq Q$.

Problem 5. Let M be a compact, oriented, smooth manifold with boundary ∂M. Prove that there does not exist a map $F : M \to \partial M$ such that $F_{|\partial M}\colon \partial M \to \partial M$ is the identity map.

Possible hint: consider the inclusion map $i\colon \partial M \to M$, and use Stokes's theorem.

Problem 6. Let $S^n \subset \mathbb{R}^{n+1}$ denote the unit sphere, and let $\omega \in \Omega^n(S^n)$ be the differential n-form given by the restriction of $x_{n+1} dx_1 \wedge \cdots \wedge dx_n \in \Omega^n(\mathbb{R}^{n+1})$ to S^n. Show that the class of ω in the de Rham cohomology $H^n_{\mathrm{dR}}(S^n)$ is nontrivial, namely that ω is closed but not exact.

Geometry and Topology Graduate Exam
Spring 2019

Solve all 6 problems. Partial credit will be given to partial solutions.

Problem 1. Let $X = S^2/\sim$ be the quotient of the sphere

$$S^2 = \big\{(x,y,z) \in \mathbb{R}^3; x^2+y^2+z^2=1\big\}$$

by the equivalence relation $\sim$ that glues together the three points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$; namely, one equivalence class of $\sim$ is equal to $\big\{(1,0,0),(0,1,0),(0,0,1)\big\}$, and all other equivalence classes consist of single points. Compute the fundamental group $\pi_1(X;x_0)$ for your preferred choice of base point $x_0 \in X$.

Problem 2.

Recall that the wedge sum $Y \vee Z$ of two spaces Y and Z, each equipped with a base point y_0 and z_0, is obtained from the disjoint union $Y \coprod Z$ by gluing $x_0 \in X$ to $y_0 \in Y$. Let $X = S^1 \vee S^2$ be the wedge sum of the circle S^1 and the sphere S^2 (for arbitrary choices of base points).

- **a.** Draw a picture of the universal cover $\widetilde{X}$ of X.
- **b.** Compute the homology group $H_2(\widetilde{X};\mathbb{Z})$, with integer coefficients.

Problem 3. Let $f\colon \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = 2z^3+3z^2$. Note that $f^{-1}\big(\{0,1\}\big) = \{-\frac{3}{2},-1,0,\frac{1}{2}\}$ (no need to check this).

- **a.** Show that the restriction $g\colon \mathbb{C}-\{-\frac{3}{2},-1,0,\frac{1}{2}\} \to \mathbb{C}-\{0,1\}$ of f is a covering map. Hint: first show that g is a local diffeomorphism.
- **b.** What is the index of the subgroup $g_*\big(\pi_1\big(\mathbb{C}-\{-\frac{3}{2},-1,0,\frac{1}{2}\};1\big)\big)$ in the fundamental group $\pi_1\big(\mathbb{C}-\{0,1\};5\big)$?

Problem 4.

Let M be a smooth m-dimensional submanifold of $\mathbb{R}^n$, and let

$$S_r^{n-1} = \big\{(x_1,\dots,x_n) \in \mathbb{R}^n; \sum_{i=1}^n x_i^2 = r^2\big\}$$

denote the sphere of radius r centered at the origin in $\mathbb{R}^n$. Show that, for every $\varepsilon > 0$, there exists an r in the interval $[1-\varepsilon, 1+\varepsilon]$ such that the intersection $M \cap S_r$ is a submanifold of M of dimension $m-1$. Possible hint: consider the map $f\colon M \to \mathbb{R}$ defined by $f(x_1,\dots,x_n) = \sum_{i=1}^n x_i^2$.

Problem 5.

Let $M = \{(x,y,z,w) \in \mathbb{R}^4; x^2+y^2+z^2-w^4=-1\}$.

- **a.** Prove that M is a differentiable submanifold of $\mathbb{R}^4$.
- **b.** Let f be the map $\mathbb{R}^4 \to \mathbb{R}$ sending $(x,y,z,w) \mapsto w$. Compute the critical values of the restriction $f_{|M}\colon M \to \mathbb{R}$. Possible hint: the tangent map $T_p f_{|M}$ of the restriction $f_{|M}$ at $p \in M$ is the restriction of $T_p f$ to $T_p M = \ker T_p g$ where $g\colon \mathbb{R}^4 \to \mathbb{R}$ is defined by $g(x,y,z,w) = x^2+y^2+z^2-w^4$.

Problem 6. Let Z be the vector field on $\mathbb{R}^2$ defined by $Z(x,y) = -y\frac{\partial}{\partial y} + 2x\frac{\partial}{\partial x}$. Compute the Lie derivative $\mathcal{L}_X(dx \wedge dy)$.

Geometry and Topology Graduate Exam
Fall 2019

Solve all seven problems. Every problem is weighted equally. Partial credit will be given to partial solutions.

Problem 1. Let X be the union of the twelve edges of a regular cube in $\mathbb{R}^3$. Compute the fundamental group of the complement $\mathbb{R}^3 - X$.

Problem 2. Let $\mathbb{CP}^1$ denote the complex projective space of dimension 1, and let $f\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ be the map induced by the polynomial

$$P(X) = X^7 + 5X^3 - 6X^2 + 1.$$

(So, f sends the line passing through $(x,1) \in \mathbb{C}^2$ to the line passing through $(P(x),1)$.) If $\alpha \in \Omega^2(\mathbb{CP}^1)$ is a differential form of degree 2 and we write

$$K := \int_{\mathbb{CP}^1} \alpha,$$

compute the integral

$$\int_{\mathbb{CP}^1} \Omega^2(f)(\alpha)$$

of the pullback $\Omega^2(f)(\alpha) \in \Omega^2(\mathbb{CP}^1)$ of α under f in terms of K. (Possible hint: degree.)

Problem 3. Let X and Y be two topological spaces and let $f,\, g : X \to Y$ be two continuous maps. Consider the topological space

$$Z = \left(Y \bigsqcup (X \times [0,1])\right) \Big/ \begin{array}{l}(x,0) \sim f(x)\\ (x,1) \sim g(x)\end{array}$$

obtained from the disjoint union $Y \bigsqcup (X \times [0,1])$ by identifying $(x,0) \sim f(x)$ and $(x,1) \sim g(x)$ for all $x \in X$. Show that there is a long exact sequence of the form

$$\cdots \longrightarrow H_{n+1}(Z) \longrightarrow H_n(X) \longrightarrow H_n(Y) \longrightarrow H_n(Z) \longrightarrow H_{n-1}(X) \longrightarrow \dots\ ,$$

and identify the homomorphisms involved.

Problem 4. Let $f\colon S^n \to S^n$ be a continuous map that has no fixed points. Find the degree of f. (Hint: $a(x) = -x$.)

Problem 5. Let $p \in \mathbb{R}[x_1, \dots, x_n]$ be a nonzero polynomial over $\mathbb{R}$ in n variables that is homogenous of degree d (i.e. $p(\lambda \cdot \vec{x}) = \lambda^d \cdot p(\vec{x})$ for all $\lambda \in \mathbb{R}$). Show that $p^{-1}(c)$ is a submanifold of $\mathbb{R}^n$ for all $c \neq 0$.

Problem 6. Define a differential 1-form on the cylinder

$$C = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1\} \subset \mathbb{R}^3$$

which is closed but not exact (and show that it has these properties).

Problem 7. Prove that the subset

$$M = \{(x_1,x_2,x_3,x_4,x_5) \in \mathbb{R}^5 : x_1^2 + x_2^2 + x_3^2 = x_4^3 + x_5^3\}$$

is not a submanifold of $\mathbb{R}^5$. (Hint: $(0,0,0,0,0)$.)

Geometry and Topology Graduate Exam
Spring 2020

Solve as many problems as you can. Partial credit will be given to partial solutions.

Problem 1. Show that the subset

$$X = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5; x_1^4 + x_2^4 = 1 + x_3^2 + x_4^2 + x_5^2\} \subset \mathbb{R}^5$$

is an orientable manifold of dimension 4. Make sure that you justify the word "orientable".

Problem 2. Let M be a compact orientable manifold with boundary. Show that there is no differentiable map $f\colon M \to \partial M$ such that $f(x) = x$ for every $x \in \partial M$.

Possible hint: Consider a volume form on ∂M.

Problem 3. Let M be an m-dimensional submanifold of $\mathbb{R}^{2m+2}$. Show that there is a hyperplane $H \subset \mathbb{R}^{2m+2}$ such that the restriction to M of the orthogonal projection $\pi_H\colon \mathbb{R}^n \to H$ is injective.

Possible hint: Use a suitable map $\{(x,y) \in M \times M; x \neq y\} \to S^{2m+1}$.

Problem 4. Consider the space X obtained from the cylinder $S^1 \times [0,1]$ by identifying the antipodal points of the circle $S^1 \times \{0\}$ and identifying the antipodal points of the circle $S^1 \times \{1\}$. (Recall that the antipodal point of $(x,y) \in S^1 \subset \mathbb{R}^2$ is the point $(-x,-y)$.) Compute the fundamental group of X.

Problem 5. Use the Mayer–Vietoris exact sequence to compute, for any space X, the homology groups $H_p(X \times S^n)$ in terms of the homology groups $H_q(X)$ (for any coefficients).

Problem 6. Let $X = S^1 \vee S^1$ be the wedge sum of two circles, namely the union of two circles meeting in exactly one point x_0. Let a_1, a_2 be the generators of $\pi_1(X; x_0)$ represented by loops going around the first and second circles, respectively. By covering space theory, for every subgroup H of $\pi_1(X; x_0)$ there is a connected covering space $p\colon \widetilde{X} \to X$ such that the image of $p_*\colon \pi_1(\widetilde{X}; \widetilde{x}_0) \to \pi_1(X; x_0)$ is equal to H. Draw a picture of $\widetilde{X}$ when H is the subgroup generated by the element $a_1 a_2 a_1 a_2 \in \pi_1(X; x_0)$.

Problem 7. For the n-dimensional sphere S^n, consider the inclusion map $i\colon S^n \to \mathbb{R}^{n+1} - \{0\}$ and a closed differential form $\omega \in \Omega^n(\mathbb{R}^{n+1} - \{0\})$. Let M be a compact oriented n-dimensional manifold without boundary and, for a differentiable map $f\colon M \to \mathbb{R}^{n+1} - \{0\}$, let $f^*(\omega) = \Omega^n(f)(\omega) \in \Omega^n(M)$ denote the pullback of ω under f. Show that the integral $\int_M f^*(\omega)$ is an integer multiple of $\int_{S^n} i^*(\omega)$.

Possible hint: Consider the radial projection $p\colon \mathbb{R}^{n+1} - \{0\} \to S^n$.

Geometry and Topology Graduate Exam
Fall 2020

Solve as many problems as you can. Partial credit will be given to partial solutions.

Problem 1. Suppose that the space X admits a universal covering space $\widetilde{X}$, namely a covering space that is path connected and simply connected. Show that, if $\widetilde{X}$ is compact, the fundamental group of X is finite.

Problem 2. Let X be the topological space obtained from a regular $2n$-gon by identifying opposite edges with parallel orientations. Write a presentation for its fundamental group $\pi_1(X; x_0)$, for a base point $x_0 \in X$ of your choice. (The answer, as a function of n, may depend on the parity of n.)

Problem 3. In $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$, consider the subset

$$X = \left(S^1 \times S^1\right) \cup \left(\{x_0\} \times B^2\right) \cup \left(B^2 \times \{x_0\}\right)$$

where the disk B^2 is bounded by the unit circle S^1 in $\mathbb{R}^2$, and where $x_0 \in S^1$. Compute the homology group $H_p(X; \mathbb{Z})$ for all p.

Problem 4. In the vector space $\mathrm{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ of all n-by-n matrices, let $\mathrm{SL}_n(\mathbb{R})$ be the special linear group consisting of all $A \in \mathrm{M}_n(\mathbb{R})$ with $\det A = 1$. For $A \in \mathrm{SL}_n(\mathbb{R})$, describe the tangent space $T_A\mathrm{SL}_n(\mathbb{R}) \subset \mathrm{M}_n(\mathbb{R})$ by an explicit equation. Possible hint: begin with the case where A is the identity matrix I_n.

Problem 5. Consider the 1–form $\lambda \in \Omega^1(M)$ and the 2-form $\omega = d\lambda \in \Omega^2(M)$ on the manifold M. Suppose that L is a submanifold of M and that, for the inclusion map $i\colon L \to M$, the pull-back $i^*(\lambda) \in \Omega^1(L)$ is exact, in the sense that there exists a function $\varphi\colon L \to \mathbb{R}$ such that $i^*(\lambda) = d\varphi$. Show that, for the unit disk $D^2 \subset \mathbb{R}^2$ and for any smooth map $f\colon D^2 \to M$ which sends the boundary of the disc to L,

$$\int_{D^2} f^*(\omega) = 0.$$

Problem 6. Let $f\colon S^1 \times S^1 \to S^1 \times S^1$ be the map that, identifying S^1 with the unit circle in the complex plane $\mathbb{C}$, is defined by

$$f(z_1, z_2) = \left(z_1^2 z_2, z_1^{-1} z_2\right)$$

for every $z_1, z_2 \in S^1 \subset \mathbb{C}$. Compute the homomorphism $H^2(f)\colon H^2_{\mathrm{dR}}(S^1 \times S^1) \to H^2_{\mathrm{dR}}(S^1 \times S^1)$ induced by f on the de Rham cohomology space $H^2_{\mathrm{dR}}(S^1 \times S^1)$. Hint: what is the degree of f?

Problem 7. Let $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ be a 2-form on $\mathbb{R}^4$ with standard coordinates x_1, x_2, x_3, x_4. Consider the vector field $Z = 3x_1\partial_{x_1} + 3x_2\partial_{x_2} + 3x_3\partial_{x_3} + 3x_4\partial_{x_4}$, and let $(\varphi_t)_{t\in\mathbb{R}}$ be the flow that it defines; you may take for granted that this flow exists for all time t. Calculate the pull back $(\varphi_t)^*\omega$. Hint: look at the differential equation that $(\varphi_t)^*\omega$ satisfies.

Geometry and Topology Graduate Exam
Spring 2021

Solve as many problems as you can. Partial credit will be given to partial solutions.

Problem 1. Let $X = S^1 \times S^1 - \{p, q\}$, with $p \neq q$, be the twice punctured 2–dimensional torus.

(1) Compute the homology groups $H_n(X, \mathbb{Z})$.
(2) Compute the fundamental group of X.

Problem 2. Let X be the figure eight, union of two circles meeting in exactly one point x_0. Recall that the fundamental group $\pi_1(X; x_0)$ is the free group on two generators a and b, respectively going once around the first and the second circle. Draw a covering $p \colon \widetilde{X} \to X$ such that $\widetilde{X}$ is connected and $p_*(\pi_1(\widetilde{X}; \widetilde{x}_0))$ is the subgroup $G \subset \pi_1(X; x_0)$ generated by the subset $\{a^2, b^2, aba, bab\}$.

Use this construction to decide whether this subgroup G is normal or not.

Problem 3. Let M be a differentiable (not necessarily orientable) manifold. Show that its cotangent bundle

$$T^*M = \{(x, u); x \in M \text{ and } u \colon T_xM \to \mathbb{R} \text{ linear}\}$$

is a manifold, and is orientable.

Problem 4.

Show that, if a map $f : S^n \to S^n$ has no fixed points, then its degree is equal to $(-1)^{n+1}$. Possible hint: Show that f is homotopic to a simple map.

Problem 5. Let T be the torus in $\mathbb{R}^3$ obtained by revolving the circle

$$\{(x, y, z) \in \mathbb{R}^3; (x-2)^2 + y^2 = 1 \text{ and } z = 0\}$$

around the y-axis. Compute the integral

$$\int_T x dy \wedge dz - y dx \wedge dz + z dx \wedge dy.$$

Problem 6. Let $f : M \to N$ be a differentiable map between two connected compact orientable manifolds of the same dimension n. Suppose that there exists a nonempty open subset U such that $f^{-1}(U)$ can be written as a disjoint union $U_1 \coprod U_2 \coprod U_3$ for which each restriction $f|_{U_i} : U_i \to U$ is a diffeomorphism. Show that f is necessarily surjective.

Problem 7. Consider the differential 2–form $\omega = \frac{dx \wedge dy}{x^2+y^2}$ on $X = \mathbb{R}^2 - \{0\}$, and denote by $Y = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = 1\}$ the unit circle inside X. Prove that, for the unit disk D^2 and for any smooth map $f \colon D^2 \to X$ which sends the boundary of the disc to Y,

$$\int_{D^2} f^*(\omega) = 0,$$

where $f^*(\omega) \in \Omega^2(D^2)$ (also denoted as $\Omega^2(f)(\omega)$) is the pull back of ω under f.

Geometry and Topology Graduate Exam
Fall 2021

Solve as many problems as you can. Partial credit will be given to partial solutions.

Problem 1. Find all of the 2-sheeted covering spaces (connected or disconnected) of $S^1 \times S^1$, up to isomorphism of covering spaces without basepoints.

Problem 2. Let $f : \mathbb{R}^4 \to \mathbb{R}$ be the function defined by

$$f(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 - x_3^2 - x_4^2.$$

(a) Find a real number r such that $f^{-1}(r)$ is a smooth manifold and prove it.
(b) Find a real number r such that $f^{-1}(r)$ is not a smooth manifold and prove it.

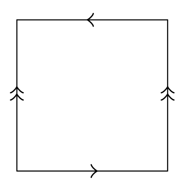
Problem 3. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere, and $i : S^2 \to \mathbb{R}^3$ be the inclusion. Compute the integral over S^2 of the restriction

$$\int_{S^2} \omega = \int_{S^2} i^*\omega$$

of the 2-form on $\mathbb{R}^3$ given by $\omega = 2x^2 dx \wedge dz - xdy \wedge dz + 3ydx \wedge dz$.

Problem 4. Let $\mathcal{D}$ be the distribution on $\mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R} = \{(x, y, z) \in \mathbb{R}^3 | y > 0\}$ given by the kernel of the 1-fom $\alpha = dz - \log(y)dx$. Is $\mathcal{D}$ integrable? Provide justification.

Problem 5. Let K be the Klein bottle (the closed square with boundary identifications as pictured below).



(a) Let $p \in K$ be the image of some point in the interior of the closed square (under the identifications above). Say whether the following assertion is true or false (and give justification): $K \backslash \{p\}$ is homotopy equivalent to $S^1 \vee S^1$.
(b) Show that K is homeomorphic to the disjoint union of two Möbius bands with the boundary circles identified.
(c) Use part (b) (whether or not you solved it) to compute $\pi_1(K)$ via van Kampen's theorem and the integral singular homology $H_*(K; \mathbb{Z})$ via the Mayer-Vietoris long-exact sequence.

Problem 6. A *space-filling curve* is a continuous surjective map $f : \mathbb{R} \to \mathbb{R}^2$ (it is a classical fact that such curves exist).

(a) Prove that if f is any such space-filling curve, then f cannot be smooth. Equivalently, prove that if $f : \mathbb{R} \to \mathbb{R}^2$ is any smooth map, then f cannot be surjective.
(b) Prove that if f is any space-filling curve, then f cannot be a homeomorphism.

Problem 7. Let X be the space given by taking the circle S^1 and attaching two 2-cells to S^1 along degree 9 and 12 attaching maps respectively, and then identifying a point in the interior of the first 2-cell with a point in the interior of the second 2-cell. Compute the integral homology $H_*(X;\mathbb{Z})$ in every degree.

Geometry and Topology Graduate Exam
Spring 2022

Solve as many problems as you can. Partial credit will be given to partial solutions.

Problem 1. Let B^3 be the solid ball of radius 2 centered at the origin in $\mathbb{R}^3$. Let S^1 be the unit circle in the xy-plane. Compute the homology groups of $B^3 - S^1$.

Problem 2. Prove that the fundamental group and the homology groups (in every degree) of $\mathbb{R}P^3$ and $\mathbb{R}P^2 \vee S^3$ are isomorphic, but that the homology groups of their universal covering spaces are not.

Problem 3. Consider the vector fields $v_k = x^k \frac{\partial}{\partial x}$ on $\mathbb{R}$, where $k \geq 0$.

(1) Find the Lie bracket $[v_i, v_j]$.
(2) Do the flows of v_i and v_j commute?
(3) Find the flow of v_2.
(4) Is v_2 complete on $\mathbb{R}$, i.e. does every one of its flow curves exist for all time?

Problem 4. (*intersecting with a plane and a cylinder*) Let M^m be any submanifold of $\mathbb{R}^N$ with $N \geq 3$. Show that there exist real numbers $a, b > 0$ so that the intersection $M \cap \{x_N = a\} \cap \{x_1^2 + x_2^2 = b^2\}$ is a submanifold (of M and hence $\mathbb{R}^N$) of dimension $m - 2$. **Note**: The condition $N \geq 3$ is not strictly necessary, it is imposed simply to ensure that x_N, x_1, and x_2 are all different coordinates to avoid the case $\{x_N = a\} \cap \{x_1^2 + x_2^2 = b^2\}$ is empty (in which case the result would be vacuously still true).

Problem 5.

Say whether each assertion is true or false, with complete justification.

(1) $S^3 \setminus \{3 \text{ points}\}$ is homotopy equivalent to $S^2 \setminus \{2 \text{ points}\}$.
(2) Any continuous map $S^2 \to S^1 \times S^1$ is null homotopic.

Problem 6. Let $f : M^n \to N^n$ be a smooth map between compact oriented manifolds of the same dimension $n > 1$, and suppose f factors as $M^n \to S^1 \times \mathbb{R}^{n-1} \to N^n$ (with $n - 1 > 0$). Show that in any neighborhood U of any point $x \in N^n$, there exists a point $y \in U$ with $f^{-1}(y)$ having an even number of points.

Problem 7. Let M^m be a manifold and $\lambda \in \Omega^1(M)$ a closed 1-form ($d\lambda = 0$). Suppose $f : S^2 \to M$ is a smooth map, where S^2 is the standard unit 2-sphere in $\mathbb{R}^3$, and let $f|_{S^1}$ denote the restriction of f to the equator circle $S^1 = \{z = 0\} \cap S^2 \subset S^2$. Show that

$$\int_{S^1} (f|_{S^1})^* \lambda = 0.$$

Geometry and Topology Graduate Exam
Fall 2022

Solve as many problems as you can. Partial credit will be given to partial solutions.

Problem 1. Consider the vector fields $X = e^x \partial_x$ and $Y = \partial_y$ on $\mathbb{R}^2$. Find all vector fields Z on $\mathbb{R}^2$ such that $[X, Z] = [Y, Z] = 0$.

Problem 2. Let X be a path-connected topological space, and let $x \in X$. Show that $\pi_1(X, x)$ is trivial if and only if, for any $x_1, x_2 \in X$, any two paths $\gamma, \delta\colon [0, 1] \to X$ from x_1 to x_2 are homotopic (through paths from x_1 to x_2).

Problem 3. Let $\mathbb{T}^2 = \mathrm{S}^1 \times \mathrm{S}^1$ be the 2-torus, and let $\alpha, \beta, \gamma \in \Omega^1(\mathbb{T}^2)$ be closed 1-forms on $\mathbb{T}^2$. Show that there exist real numbers $a, b, c \in \mathbb{R}$ such that $a\alpha + b\beta + c\gamma$ is exact.

Problem 4. Let ω be a 1-form on the sphere $S^2 = \{(x, y, z)|x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$. Show that if ω is invariant under rotations, i.e. $\phi^*\omega = \omega$ for all $\phi \in SO(3)$, then $\omega = 0$.

Problem 5. Show that if M and N are compact, connected smooth manifolds, then every submersion $f\colon M \to N$ is surjective.

Problem 6. Let X be the complement of a point in $S^1 \times S^1 \times S^1$. Calculate the fundamental group and homology groups of X.

Problem 7. Show that every continuous map from $\mathbb{RP}^2 \times \mathbb{RP}^2$ to $S^1 \times S^1 \times S^1 \times S^1$ is null-homotopic.

Geometry and Topology Graduate Exam
Spring 2023

Solve as many problems as you can. Partial credit will be given to partial solutions.

Problem 1. Let X be a Hausdorff topological space, and let $\pi : \widetilde{X} \to X$ be its universal cover, i.e. $\widetilde{X}$ is path connected and simply connected and π is a covering map. Prove that if $\widetilde{X}$ is compact then the fundamental group of X is finite.

Problem 2. Let A be an $n \times n$ matrix which is symmetric and nonsingular, and let c be a nonzero real number. Prove that

$$\{x \in \mathbb{R}^n \mid \langle Ax, x\rangle = c\}$$

is a smooth submanifold of $\mathbb{R}^n$, and state its dimension. Here $\langle -, -\rangle$ denotes the standard inner product on $\mathbb{R}^n$.

Problem 3. Let $\omega \in \Omega^2(M)$ be an exact 2-form on a manifold M. Prove that for any map $f\colon S \to M$ from a closed orientable surface (i.e., closed orientable 2-dimensional manifold) S, there must be some $p \in S$ such that $(f^*\omega)_p = 0$.

Problem 4. Let $\mathbb{T}^2 = S^1 \times S^1$ denote the standard 2-torus and S^2 the standard 2-sphere. Let X be the space obtained by identifying two distinct points a_1, a_2 from $\mathbb{T}^2$ to some point $p \in S^2$. Compute (1) the (integral) homology groups of X in every degree, and (2) the fundamental group of X.

Problem 5. Let $n > 1$, let $\mathbb{T}^n = (\mathrm{S}^1)^n$ denote the n-torus, and let S^n denote the standard unit sphere in $\mathbb{R}^{n+1}$.

(a) Let $f : \mathbb{T}^n \to S^n$ be a smooth map satisfying the following properties:
- there exists 5 mutually disjoint open subsets $U_1, \dots, U_5$ of $\mathbb{T}^n$ such that for each i $f|_{U_i}$ is a diffeomorphism from U_i onto the open southern hemisphere $S^n \cap \{x_{n+1} < 0\}$;
- The image of the complement of these subsets lies in the northern hemisphere; that is $f(\mathbb{T}^n - \cup_i U_i) \subset S^n \cap \{x_{n+1} \geq 0\}$.

(You may take for granted such an f exists). Show that the induced map on nth de Rham cohomology $f^* : H^n_{dR}(\mathrm{S}^n) \to H^n_{dR}(\mathbb{T}^n)$ must be non-zero.

(b) Show that there does *not* exist a continuous map $f\colon \mathrm{S}^n \to \mathbb{T}^n$ from the n-sphere to the n-torus $\mathbb{T}^n = (\mathrm{S}^1)^n$ inducing a non-zero map $f_*\colon \mathrm{H}_n(\mathrm{S}^n) \to \mathrm{H}_n(\mathbb{T}^n)$ of n-th homology groups.

Problem 6. Let X be a topological space. Suppose for some k that we can cover X by k open sets $U_1, \dots, U_k$ so that each U_i is contractible as is each higher intersection of s open sets $U_{i_1} \cap \cdots \cap U_{i_s}$ for every s. Prove that the reduced homology $\tilde{H}_i(X) = 0$ for all $i \geq k - 1$.

Problem 7. Let X denote the vector field on $\mathbb{R}^3$ given in standard coordinates by $X = x_1 \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_2} + 3x_3 \frac{\partial}{\partial x_3}$, and let $\phi_t : \mathbb{R}^3 \to \mathbb{R}^3$ denote the induced flow (you may take for granted that this exists for all time and is an oriented diffeomorphism).

If $R = [0,1]^3$ denotes the unit cube, compute the rate of change of the (standard) volume of $\phi_t(R)$ at $t = 0$. That is, compute:

$$\frac{d}{dt}(\int_{\phi_t(R)} dx_1 dx_2 dx_3)_{t=0}.$$

Hint: Re-express the above integral as an integral of a form which is varying in t, over a region that is not varying in t. You may also use the fact that $\frac{d}{dt}\int_A(\omega_t) = \int_A \frac{d}{dt}(\omega_t)$.

Geometry and Topology Graduate Exam
Fall 2023

Solve as many problems as you can. Partial credit will be given to partial solutions.

Problem 1. Show that for any $n \geq 1$, the subset $\mathrm{SL}_n(\mathbb{R}) \subset \mathbb{R}^{n \times n}$ consisting of matrices of determinant 1 is a smooth submanifold.

Problem 2. Let $\alpha \in \Omega^2(\mathbb{R}^3)$ be the 2-form defined by

$$\alpha = (x^3 + y + z)\,\mathrm{d}y \wedge \mathrm{d}z - (x + y^3 + z)\,\mathrm{d}x \wedge \mathrm{d}z + (x + y + z^3)\,\mathrm{d}x \wedge \mathrm{d}y.$$

Compute the integral

$$\int_{\mathrm{S}^2} \alpha$$

over the unit sphere $\mathrm{S}^2 \subset \mathbb{R}^3$ (endowed with a fixed orientation of your choosing).
Hint: Recall that, for the spherical coordinates (r, θ, φ) *with* $x = r\cos\theta\cos\varphi$, $y = r\sin\theta\cos\varphi$, *and* $z = r\sin\varphi$, *we have* $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = r^2 \cos\varphi\,\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\varphi$.

Problem 3. Let X be a manifold with $\pi_2(X, x) = 0$ for all $x \in X$. Is it necessarily the case that $\mathrm{H}_2(X; \mathbb{Z}) = 0$ as well? Explain.

Problem 4. What are the integral homology groups of $S^1 \vee S^2 \vee S^3 \vee S^4$?

Problem 5. Let V be a smooth vector-field on a closed manifold M and let $\varphi : \mathbb{R} \times M \to M$ be the flow generated by V. Consider the quotient space X by the flow, i.e.

$$X = M/{\sim} \qquad \text{where} \qquad p \sim q \text{ if and only if } q = \varphi(t, p) \text{ for some time } t.$$

Prove or give a counter-example to the following statements.

(a) X is compact.

(b) X is a closed manifold.

Problem 6. Give an example of a covering space $X \to Y$ which is not a regular covering space.

Problem 7. Show that the Cantor set does not admit a CW complex structure.

Geometry and Topology Screening Exam
Spring 2024

Instructions: Solve as many of the following 7 problems as you can. Solutions will be graded for correctness, completeness and clarity. Partial credit will be awarded for partial solutions indicating clear progress.

Problem 1. Let M be a path-connected smooth manifold. Prove that for any $p_1, p_2 \in M$, there is a diffeomorphism φ of M with $\varphi(p_1) = p_2$.

Problem 2. Show that for any vector bundle $p\colon E \to B$ over a space B, the map p is a homotopy equivalence.

Problem 3. Let X be a closed manifold, and let θ be a degree 1 de Rham class such that

$$\int_\Gamma \theta \in \mathbb{Z} \qquad \text{for any cycle } \Gamma.$$

Show that there exists a map $\varphi\colon X \to S^1$ so that the induced map

$$\varphi^*\colon \mathrm{H}^1_{\mathrm{dR}}(S^1) \to \mathrm{H}^1_{\mathrm{dR}}(X)$$

contains θ in its image.

Problem 4. Suppose $f\colon M \to N$ is a smooth map between connected smooth manifolds which is both an immersion and a submersion. Is f necessarily a diffeomorphism? Prove it or provide a counter-example.

Problem 5. Let 0_n be the $n \times n$ 0-matrix and let I_n be the $n \times n$ identity matrix. Let J be the $2n \times 2n$ matrix given by

$$J = \begin{bmatrix} 0_n & -I_n \\ I_n & 0_n \end{bmatrix}$$

Finally, let $G \subset \mathrm{GL}_{2n}(\mathbb{R})$ be the group of invertible $2n \times 2n$ matrices A that commute with J. Show that G is a sub-manifold of $\mathrm{GL}_{2n}(\mathbb{R})$ and compute its dimension.

Problem 6. Let X be a closed smooth manifold and let V be a vector-field. Suppose that α and β are closed 1-forms on X such that $\alpha(V)$ and $\beta(V)$ are constant functions. Show that

$$\alpha \wedge \beta$$

is invariant under the flow generated by V.

Problem 7. Let $K \subset S^3$ be a smoothly embedded S^1. Compute the first homology group of $S^3 \setminus K$ over $\mathbb{Z}$.