REAL ANALYSIS GRADUATE EXAM

Spring 2024

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Consider a sequence $f_n = f_n(x), x \geq 0, n = 1, 2, \ldots,$ of continuously differentiable functions with corresponding derivatives f_n'. Assume that $f_n(0) = 0$ and

$$\lim_{n\to\infty} \int_0^{+\infty} |f_n'(x)|^2 \, dx = 0.$$

Is is true of false that

$$\lim_{n\to\infty} \sup_{x\geq 0} |f_n(x)| = 0?$$

Explain your conclusion by proof or counterexample.

2. Let $(X, \mathcal{A}, \mu)$ be a complete measure space and let f be a non-negative integrable function on X. Put $b(t) = \mu(\{x \in X : f(x) \geq t\})$. Show that

$$\int_X f \, d\mu = \int_0^\infty b(t) \, dt.$$

3. Let X be a compact metric space, μ a finite non-negative Borel measure on X. Suppose $\mu(\{x\}) = 0$ for all $x \in X$. Prove that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mu(X) < \varepsilon$ whenever E is a Borel set in X of diameter $< \delta$.

4. (i) Assume that $f \in \mathscr{L}^1(0, \infty)$. Prove that

$$g(x) = \int_0^\infty \frac{f(y)}{x+y} \, dy$$

is differentiable at every $x > 0$.
(ii) In (i) find an example of $f \in \mathscr{L}^1(0, \infty)$ such that $g : [0, \infty) \to \mathbb{R}$ is not differentiable at $x = 0$.