COMPLEX ANALYSIS GRADUATE EXAM

Spring 2024

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let $n \in \mathbb{N}$ and $\vartheta \in (0, \pi)$. Prove that

$$\frac{1}{2\pi i} \int_{|z|=2} \frac{z^n}{1 - 2z\cos\vartheta + z^2}\, dz = \frac{\sin(n\vartheta)}{\sin\vartheta},$$

where the integration contour is positively (counter clockwise) oriented.

2. Describe all the analytic functions f in $\mathbb{C} \setminus \{0\}$ with the property

$$|f(z)| \leq |z|^2 + \frac{1}{\sqrt{|z|}}$$

for all $0 < |z| < \infty$.

3. Assume $f_n, n = 1, 2, \ldots,$ is analytic in $\mathbb{D} = \{|z| < 1\}$ and satisfies $|f_n| \leq 10$. Assume also that the limits $\lim_{n\to\infty} f_n(2^{-j})$ exist for every $j = 1, 2, \ldots$. Prove that the limits $\lim_{n\to\infty} f_n(z)$ exist for all $z \in \mathbb{D}$.

4. Consider the function

$$f(z) = \sum_{k=0}^{\infty} z^{2^k} = z + z^2 + z^4 + z^8 + z^{16} + \cdots .$$

Explain why f is analytic in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ and has the unit circle $\{z : |z| = 1\}$ as natural boundary, that is, cannot be continued analytically outside of $\mathbb{D}$.