Algebra qualifying exam, January 2024

Justify all arguments completely. Every ring R is assumed to have a unit $1 \in R$. Reference specific results whenever possible.

1. Let G be a simple group of order 168. Show that G is isomorphic to a subgroup of A_8, the alternating group of degree 8. Show that G is not isomorphic to a subgroup of A_6.

2. Let K be a field and A be a finite-dimensional, semisimple K-algebra. Let $Z(A)$ denote the center of A. Prove that two finitely-generated A-modules M and M' are isomorphic as A-modules if and only if they are isomorphic as $Z(A)$-modules.

3. Let f and g be polynomials in $\mathbb{C}[x_1, \ldots, x_{24}]$. Suppose that for each value $z \in \mathbb{C}^{24}$ at which $f(z) = 0$, we also have $g(z) = 0$. Prove that f divides some power of g.

4. Define the Jacobson radical of a ring to be the intersection of all maximal left ideals of this ring. Let $\phi : R \to S$ be a surjective morphism of rings. Prove that the image by ϕ of the Jacobson radical of R is contained in the Jacobson radical of S.

5. Construct an example (or merely prove the existence) of a 10×10 matrix over $\mathbb{R}$ with minimal polynomial $(x+1)^2(x^4+1)$ which is not similar to a matrix over $\mathbb{Q}$.

6. Let F be a field of characteristic not 2. Show that if $f(x) = x^8 + ax^4 + bx^2 + c$ is an irreducible polynomial over F for some $a, b, c \in F$, then the Galois group of the splitting field of f is solvable.