MATH 507a **QUALIFYING EXAM** **January 2024**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems.

(1)(a) Suppose $X \leq 1$ a.s. If $EX = 1$, show that $X = 1$ a.s. (Write a proof, don't just cite a theorem.)

(b) If Y has characteristic function $\varphi(t)$ and $\varphi(t) = 1$ for some $t \neq 0$, show that Y is a discrete r.v., in fact show there exists $a > 0$ such that Y/a is integer-valued. HINT: Use (a).

(2) Let $X_1, X_2, \dots$ be i.i.d. nonnegative random variables with $EX_1 < \infty$. Let $S_n = \sum_{i=1}^n X_i$.

(a) Show that $X_n/n \to 0$ a.s.

(b) Show that $\max(X_1, \dots, X_n)/n \to 0$ a.s. HINT:

$$\max(X_1, \dots, X_n) \leq \max(X_1, \dots, X_k) + \max(X_{k+1}, \dots, X_n),$$

for all $k \leq n$. Use (a).

(c) Show that

$$\max_{1 \leq k \leq n} \frac{X_k}{S_n + 1} \to 0 \quad a.s.$$

(3)(a) Suppose $0 \leq X_1 \leq X_2 \leq \dots$ are random variables with $E(X_n^2) < \infty$, $EX_n \to \infty$, and $\mathrm{var}(X_n)/(EX_n)^2 \to 0$. Show that $X_n \to \infty$ a.s.

(b) Let $\{A_n\}$ and $\{B_n\}$ be events such that

(i) the events B_n are mutually independent,

(ii) $P(A_n) \to 1$,

(iii) B_n is independent of A_n for each n,

(iv) $\sum_n P(B_n) = \infty$.

Show that $P(A_n \cap B_n \text{ i.o.}) = 1$. HINT: Instead of Borel–Cantelli, consider properties of $S_n = \sum_{k=1}^n 1_{A_k \cap B_k}$. Part (a) may be useful. Also, $P(A_j \cap B_j \cap A_k \cap B_k) \leq P(B_j \cap B_k)$.