MATH 507a **QUALIFYING EXAM** **February 1, 2012**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway.

1. Let $X_1, X_2, \ldots$ be independent Poisson distributed random variables, with $\lambda_i = \mathbb{E}\, X_i$. Show that
a) If $\sum_i \lambda_i < \infty$, then $\sum_i X_i$ converges, almost surely, to a finite limit.
b) If $\sum_i \lambda_i = \infty$, then $\sum_i X_i \; = \infty$ almost surely.

2. Let $X_1, X_2, \cdots$ be i.i.d. with uniform distribution on $[-1, 1]$. Find the limit distribution of

$$Y_n := \frac{\sum_{i=1}^n X_i}{\sqrt{\sum_{i=1}^n |X_i|^2}}.$$

3. Let $p \in (0, 2)$ and let $\xi_n,\ n \geq 1$, be iid random variables. Show that the following two conditions are equivalent:

(a) With probability one, the limit

$$\lim_{n\to\infty} \frac{1}{n^{1/p}} \sum_{k=1}^{n} \xi_k$$

exists and is finite.

(b) $E|\xi_1|^p < \infty$ AND either $E\xi = 0$ or $p \leq 1$.

Math 507a 2012 Fall Qualifying Exam

1. Let $X_n, n \geq 1$ be random variables such that X_n has probability density function

$$f_n(x) = \begin{cases} 1 + \sin(2\pi n x) & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that the sequence $\{X_n,\ n \geq 1\}$ converges in distribution as $n \to \infty$ and identify the limit.

(b) Assume in addition that X_n are independent. Find a sequence a_n of real numbers such that with probability one,

$$0 < \limsup_{n\to\infty} (\log X_n)/a_n < \infty.$$

If you are unable to do b) as stated, replace X_n with U_n, i.i.d. variables with the uniform distribution on $[0, 1]$.

2. A random variable X is called infinitely divisible if, for every $n = 1, 2, 3, \ldots$ there exist independent and identically distributed random variables $X_1^{(n)}, \ldots, X_n^{(n)}$ such that the distribution of $X_1^{(n)} + \cdots + X_n^{(n)}$ is that of X.

a. Suppose that X_m is a sequence of infinitely divisible random variables with $\sup_m \mathrm{Var}(X_m) < \infty$, and that X_m converges to X in distribution. Prove that X is infinitely divisible. Hint: Use tightness.

b. Let X be a random variable with the probability density function

$$f(x) = \begin{cases} 1 - |x| & \text{if } -1 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Show X is not infinitely divisible. Hint: Consider that X has bounded support.

c. Let U be a random variable uniformly distributed over the interval $[-1/2, 1/2]$. Show U is not infinitely divisible. (You may use part b.)

3. Suppose μ is a nonnegative, sigma-finite measure on $(0, \infty)$ such that

$$m := \int_0^\infty x \, \mu(dx) \in (0, \infty).$$

Let $i = \sqrt{-1}$ and let ϕ be given by

$$\phi(u) := \exp\left(\int (e^{iux} - 1) \, \mu(dx) \right).$$

a) Prove that there is a random variable T such that $Ee^{iuT} = \phi(u)$ for all real u. Hint: With δ_x point mass at $x > 0$ and $\lambda > 0$, consider the special cases $\mu = \lambda\delta_1$, $\mu = \lambda\delta_x$, then $\mu = \sum_{i=1}^n \lambda_i \delta_{x_i}$.

b) Assuming a), prove $m = ET$.

c) Assuming a), give a simple integral expression for the variance of T.

Math 507a 2013 Spring Qualifying Exam

1. For each $n = 1, 2, \ldots$, let $X_{1,n}, \ldots, X_{n,n}$ be independent mean zero random variables with variances $\sigma_{i,n}^2 = \mathrm{Var}(X_{i,n})$ satisfying $\sum_{i=1}^n \sigma_{i,n}^2 = 1$. Recall that the Lindeberg-Feller Central limit theorem says that the sum $W_n = \sum_{i=1}^n X_{i,n}$ converges in distribution to the standard normal $\mathcal{N}(0,1)$ whenever the Lindeberg condition holds, that is, when

$$\lim_{n\to\infty} L_{n,\epsilon} = 0 \quad \text{for all } \epsilon > 0, \quad \text{where} \quad L_{n,\epsilon} = \sum_{i=1}^n E\left(X_{i,n}^2 \mathbf{1}(|X_{i,n}| \geq \epsilon)\right).$$

a. Prove that the Lindeberg condition is satisfied when $X_{1,n} = X_1/\sqrt{n}, \ldots, X_{n,n} = X_n/\sqrt{n}$ and $X_i, i = 1, 2 \ldots$, are independent and identically distributed.

b. Prove that the Lindeberg condition is satisfied when

$$\lim_{n\to\infty} \sum_{i=1}^n E|X_{i,n}|^p = 0 \quad \text{for some } p > 2.$$

c. Prove that the Lindeberg condition is not necessary for the weak convergence of W_n to $\mathcal{N}(0,1)$.

2. Let X be a random variable such that $m_n = EX^n$ exists for all $n = 1, 2, \ldots$.

a. Prove that H_n, the $n \times n$ matrix with entries m_{i+j-2}, is non-negative definite.

b. Prove that if $P(X \in [0,1]) = 1$, then

$$(-1)^k \Delta^k m_n \geq 0 \quad \text{for all } n, k \geq 0,$$

where $\Delta m_n = m_{n+1} - m_n$.

3. Let $Y, Y_1, Y_2, \ldots$ be independent and identically distributed, with the unit exponential distribution, $P(Y > t) = e^{-t}$ for positive real t. Let

$$W = \gamma + \sum_{k\geq 1} \frac{Y_k - 1}{k},$$

where $\gamma = \lim_{n\to\infty}\left((1+\frac{1}{2}+\cdots+\frac{1}{n}) - \log n\right)$ is Euler's constant.

a) Compute the characteristic function of W, $\phi(u) := Ee^{iuW}$.

b) Show that $\int_{-\infty}^{\infty} |\phi(u)|\, du < \infty$.

c) Does the finiteness property in b) prove that W has an absolutely continuous distribution?

d) Find the density $f(w)$ of W. Even if you cannot simplify the integral, be sure to show the appropriate Fourier inversion formula. Hint: Consider $W_n = \gamma + \sum_{1\le k\le n} \frac{Y_k - 1}{k}$ and relate W_n to the maximum of n independent exponentially distributed random variables.

MATH 507a QUALIFYING EXAM September 23, 2013. One hour and 50 minutes, starting at 5pm.

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway.

1. (a) Let $X_1, X_2, \ldots$ be a sequence of random variables such that $|X_n| \le 1$ for all n and $\lim_{n\to\infty} X_n = X$ in probability for some random variable X. Is it true that

$$\lim_{n\to\infty} \frac{X_1 + X_2 + \cdots + X_n}{n} = X$$

in probability? Explain your reasoning.

(b) Would the conclusion be different, if the hypothesis $|X_n| \le 1$ for all n were eliminated? If your conclusion was true for (a), and false for (b), you should provide an explicit counterexample.

2. (a) Show that for a nonnegative r.v. X,

$$\sum_{n=1}^{\infty} \mathbf{P}(X \ge n) \le \mathbf{E}X \le 1 + \sum_{n=1}^{\infty} \mathbf{P}(X \ge n).$$

(b) Let X_n be a sequence of i.i.d. r.v.'s with $\mathbf{E}|X_n| = \infty$ for each n.
Show that for $k \ge 1$,

$$\sum_n \mathbf{P}(|X_n| \ge kn) = \infty,$$

and

$$\limsup_n |X_n| n^{-1} = \infty \text{ a.s.}$$

Deduce from here that

$$\limsup_n \left| \frac{X_1 + \ldots + X_n}{n} \right| = \infty \text{ a.s.}$$

3. Let X_n be a sequence of i.i.d. r.v.'s with $\mathbf{E}X_k = \mu > 0$ and $Var(X_k) = \sigma^2$. Let Y_n be a sequence of i.i.d. r.v.'s with $\mathbf{P}(Y_k = 1) = \mathbf{P}(Y_k = -1) = 1/2$. In addition assume (X_n) and (Y_n) are independent. Calculate the limiting distribution of

$$\frac{\sqrt{n}\sum_{k=1}^n X_k Y_k}{\sum_{k=1}^n X_k}$$

as $n \to \infty$. Specify the distribution fully, including name and parameters, if applicable.

MATH 507a QUALIFYING EXAM February 4, 2014. One hour and 50 minutes, starting at 5pm.

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway.

1. Let $X_1, X_2, \cdots$ be uncorrelated random variables with $E[X_i] = \mu$ and $\mathrm{Var}(X_i) \leq C < \infty$. If $S_n = X_1 + \cdots + X_n$, *prove* that as $n \to \infty$, $S_n/n \to \mu$ (part a) in L^2 and (part b) in probability. For part c), say whatever you can regarding almost sure convergence, for the sequence S_n/n — maybe something about a counterexample, or about convergence along subsequences — no need to give a proof.

2. A person plays an infinite sequence of games. He wins the nth game with probability $\frac{1}{\sqrt{n}}$, independently of the other games.

(i) Prove that for any A, the probability is 1 that the player will accumulate at least A dollars if he gets \$1 each time he wins two games in a row. (Equivalently, his total winnings W satisfies $1 = P(W = \infty)$.)

(ii) Does the claim in part (i) hold if the player gets \$1 each time he wins three games in a row?

3. Let X_n be a sequence of independent r.v. such that X_n is uniformly distributed on $[0, n^2]$. Find sequences a_n, b_n such that the sequence $\left(\sum_{i=1}^{n} X_i - a_n\right)/b_n$ converges in distribution to a nondegenerate limit. What is that limit? [There are three tasks: part a) is to name the sequences of constants, part b) is to name the nondegenerate distribution, and part c) is to prove, or at least sketch a proof, of this distributional convergence.]

MATH 507a QUALIFYING EXAM September 24, 2014. One hour and 50 minutes, starting at 4pm.

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway.

1. Let $E_1, E_2, \ldots$ be arbitrary events. Let $G := \limsup_n E_n = \{E_n \text{ i.o.}\}$.

(a) Show that $P(G) = 1$ if and only if $\sum_n \mathbb{P}(A \cap E_n) = \infty$ for all events A having $\mathbb{P}(A) > 0$.

(b) True or false: if $\mathbb{P}(G) = 0$ then $\sum_n \mathbb{P}(A \cap E_n) < \infty$ for all events A having $P(A) > 0$. If you think this is true, then provide a proof; if you think this is false, then provide a counter-example.

2. Let $\{X_n,\ n \geq 1\}$ and $\{Y_n,\ n \geq 1\}$ be two sequences of random variables such that $\lim_{n\to\infty} X_n = X$ in distribution for some random variable X with $\mathbb{P}(|X| < \infty) = 1$, and

$$\lim_{n\to\infty} \mathbb{P}(Y_n > c) = 1$$

for every $c > 0$.

Prove that

$$\lim_{n\to\infty} \mathbb{P}(X_n + Y_n > c) = 1$$

for every $c > 0$.

3. Let $a > 0$, let $X_n,\ n \geq 1$, be iid random variables that are uniform on $(0, a)$, and let $Y_n = \prod_{k=1}^{n} X_k$. Determine, with a proof, all values of a for which $\lim_{n\to\infty} Y_n = 0$ with probability one.

MATH 507a GRADUATE EXAM

FALL 2015

Answer as many questions as you can. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. If you cannot do part (a) of a problem, you can still get credit for (b), (c) etc. by assuming the answer to (a). Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Suppose $X_1, X_2, ...$ are iid with values in a bounded interval $[a, b]$ with density $f(x)$.

(a) Suppose the distribution is uniform in $[a, b]$. Show that $\liminf_n n(X_n - a) = 0$ a.s.

(b) Still supposing the distribution is uniform, what can you say about $\liminf_n n^2(X_n - a)$?

(c) Suppose instead that the endpoint $a = 0$ and the density satisfies $f(x) \sim cx^{-1/2}$ as $x \to 0$, with $0 < c < \infty$. (Here $\sim$ means the ratio converges to 1.) Find the new values of the lim infs in (a) and (b).

(2) Let ξ be a non-negative random variable. Prove that for $b > 0$,

$$\lim_{t \to +\infty} \frac{1}{t} \log \mathbb{E} e^{-t\xi} = -b$$

if and only if

$$\mathbb{P}(\xi \geq b) = 1.$$

(3) Suppose that $0 < \alpha \leq 2$, that $X, X_1, X_2, \ldots$ are independent and identically distributed, and that $S_n = X_1 + \cdots + X_n$. Assume that X has characteristic function

$$\phi(u) := \mathbb{E}(e^{iuX}) = e^{-|u|^\alpha}.$$

(a) Is X symmetric (meaning X and $-X$ have the same distribution)?

(b) Find a function $f(n)$, depending on α, such that $S_n/f(n)$ has the same distribution as X.

(c) For what values of α does S_n satisfy the conclusion of the central limit theorem, that is, $S_n/\sqrt{n}$ converges in distrubution to a normal random variable (with strictly positive variance) as $n \to \infty$?

(d) For what values of α does S_n/n converge to 0 in probability as $n \to \infty$?

MATH 507a GRADUATE EXAM

Fall 2016

Answer as many questions as you can. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. If you cannot do part (a) of a problem, you can still get credit for (b), (c) etc. by assuming the answer to (a). Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Suppose $X_1, X_2, \ldots$ are independent non-negative random variables. Show that $\sum_{n=1}^{\infty} X_n < \infty$ almost surely if and only if $\sum_{n=1}^{\infty} E[X_n/(1+X_n)] < \infty$. HINT: Consider the truncated variables $X_n' = \min(X_n, 1)$.

(2) Recall that m is a median of X if $P(X \le m) \ge 1/2$ and $P(X \ge m) \ge 1/2$.

Suppose X has density f and $E|X| < \infty$. Show that m is a median of X if and only if the function $g(a) = E(|X - a|)$ is minimized (not necessarily uniquely) at $a = m$.

(3) Let $X_1, X_2, \ldots$ be iid random variables with common density function f. Define $N_1, N_2, \ldots$ to be the times when the values X_i set a new record high, that is, $N_1 = 1, N_i = \inf\{k > N_{i-1} : X_k > X_{N_{i-1}}\}$. Let $Y_i = X_{N_{i+1}} - X_{N_i}$; this is the margin by which the record is set.

For each case (a) and (b) determine whether the sequence $\{Y_i\}$ converges in distribution as $i \to \infty$, and find the limiting distribution if it exists. HINT: Condition on X_{N_i}.

(a) $f(t) = e^{-t}$ for $t > 0$, $f(t) = 0$ for $t \le 0$.

(b) $f(t) = t^{-2}$ for $t > 1$, $f(t) = 0$ for $t \le 1$.

MATH 507a QUALIFYING EXAM Friday, February 13, 2015. One hour and 50 minutes, starting at 4:30pm.

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway.

1. Let $X_n,\ n \geq 1$, be independent random variables such that each X_n has Poisson distribution with mean n^r for some real number r. Determine the values of r for which

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} X_n}{\sum_{n=1}^{N} n^r} = 1$$

with probability one.

2. Let $X_n,\ n \geq 1$, be a sequence of iid random variables with a continuous distribution function. For $m \geq 1$, let E_m be the event that a record occurs at moment m:

$$E_1 = \Omega,\ E_m = \{X_m > X_k,\ 1 \leq k < m\},\quad m \geq 2.$$

Show that
a) $\mathbb{P}(X_m = X_n) = 0$ for $m \neq n$;
b) $P(E_n) = 1/n$;
c) E_n and E_m are independent if $m \neq n$;
d) With probability one, infinitely many records occur.

3. Let $X_n,\ n \geq 1$, be iid random variables that are uniform on $(0, 2.5)$, and let $Y_n = \prod_{k=1}^{n} X_k$. True or false: $\lim_{n\to\infty} Y_n = 0$ with probability one. Explain your conclusion.

MATH 507a GRADUATE EXAM

Spring 2016

Answer as many questions as you can. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. If you cannot do part (a) of a problem, you can still get credit for (b), (c) etc. by assuming the answer to (a). Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Let X and Y be random variables defined on the same probability space, and suppose that $X+Y$ has the same distribution as X, that is, for all t,

$$P(X+Y \leq t) = P(X \leq t). \tag{1}$$

(a) Does it follow that $P(Y=0)=1$?

(b) If we assume $Y>0$, does (1) then imply that $P(Y=0)=1$?

(c) If (instead of $Y>0$) we assume X and Y are independent, does (1) then imply that $P(Y=0)=1$?

In each case, either give a proof or construct a counterexample.

HINTS (IN NO PARTICULAR ORDER):

(i) Recall that X and $X+Y$ have the same distribution if and only if $E(f(X)) = E(f(X+Y))$ for all bounded continuous functions f on $\mathbb{R}$.

(ii) You may use the following theorem: If a random variable Z has characteristic function φ_Z satisfying $\varphi_Z(t) = 1$ for all $t \in (-\delta, \delta)$ for some $\delta > 0$, then $P(Z=0)=1$.

(iii) For one of the parts it may be useful to consider symmetric X.

(2) Let n points be i.i.d., uniformly distributed on the unit circle. Let Δ_n be the smallest distance between any two of these points. Show that $n^\theta \Delta_n \to 0$ in probability as $n \to \infty$, for all $0 < \theta < 2$. HINT: Divide the circle into small arcs and find the probability that at least one arc contains 2 or more points.

(3) Let $X_1, X_2, \ldots$ be iid with $P(X_1 > 0) = 1$. Define $S_n = X_1 + \cdots + X_n$.

(a) Suppose $E(X_1) < \infty$. Show that

$$(*) \qquad \lim_{n\to\infty} \frac{X_n}{S_n} = 0 \quad \text{a.s.}$$

(b) Construct an example in which $E(X_1) = \infty$ and $(*)$ in part (a) is false.

HINT for (b): Let X_1 take values $a_1 < a_2 < \ldots$ and let $p_n = P(X_1 = a_n)$. For each k, consider the least n for which $X_n \geq a_k$. What is the maximum possible value of S_{n-1} for this n? Use this to choose the a_k's and p_k's so that, for this sequence of n's, $(*)$ will fail.

If you can't find a specific choice of a_k's and p_k's, at least try to describe qualitatively how these sequences need to behave, for example should they grow, or approach 0, rapidly or slowly, and why.

MATH 507a PROBABILITY GRADUATE EXAM
Spring 2017

Answer as many questions as you can. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Suppose that $X_1, X_2, X_3, \ldots$ are independent and that X_n has a uniform distribution on the interval $[a_n, b_n]$ for $n \geq 1$, where $a_n < b_n$. Find necessary and sufficient conditions on the sequences $\{a_n\}$ and $\{b_n\}$ so that $\sum_n X_n$ converges almost surely.

(2) Suppose that $X_1, X_2, X_3, \ldots$ are independent and identically distributed with distribution function $F(x)$ satisfying $x(1 - F(x)) \to c$ as $x \to \infty$ for some positive constant c. Define $M_n = \max(X_1, X_2, \ldots, X_n)$.

(i) Find the distribution function of M_n in terms of F.

(ii) Show that M_n/n converges in distribution as $n \to \infty$ and find the distribution function of the limit.

(3) Suppose that $X_1, X_2, \ldots$ are i.i.d. with $0 < E(X_1^2) < \infty$, and let $S_n = X_1 + X_2 + \cdots + X_n$.

(i) Show that $\dfrac{|X_n|}{\sqrt{n}} \to 0$ a.s.

(ii) Without using the Law of the Iterated Logarithm, show that $\limsup_{n\to\infty} \dfrac{|S_n|}{\sqrt{n}} = \infty$ a.s. HINT: How does this relate to the events $\left\{ \dfrac{|S_n|}{\sqrt{n}} > c \right\}$ with $c > 0$? Also, for fixed k, do the values $X_1, \ldots, X_k$ affect the lim sup?

(4) Suppose that $\{X_n\}$ is a family of random variables such that $EX_n^2 \leq C$ for some $C < \infty$.

(i) Show that the family $\{X_n\}$ is tight.

(ii) Suppose that $X_n \Rightarrow X$. Show that $EX^2 \leq C$ and that $EX_n \to EX$. (Here $\Rightarrow$ means convergence in distribution.)

(iii) Give an example where $X_n \Rightarrow X$ but $EX_n^2 \not\to EX^2$.

MATH 507a **QUALIFYING EXAM** **January 10, 2019**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems.

1. Let $X_1, X_2, \ldots$ be independent and identically distributed (i.i.d.) random variables with density

$$f(x) = \begin{cases} |x|^{-3} & \text{if } |x| > 1 \\ 0 & \text{otherwise} \end{cases}$$

and characteristic function $\phi(t)$. Let $S_n := X_1 + \cdots + X_n$.

(a) Show that $\mathbb{E}\,(X_1^2) = \infty$.

(b) It is a fact that $1 - \phi(t) \sim t^2 \log(1/|t|)$ as $t \to 0$; you are not asked to prove this. Using this fact, propose constants $a_1, a_2, \ldots$ and prove that S_n/a_n converges in distribution to the normal distribution with mean 0 and variance 1.

2. Let $X_1, X_2, \ldots$ be independent, uniformly distributed in $(0, 1)$. Let $S_n := X_1 + \cdots + X_n$, with $S_0 = 0$. For $t > 0$ let

$$N_t = \max\{n : S_n < t\}$$

so that $N_t < n$ iff $X_1 + \cdots + X_n \geq t$.
Show that N_t/t is almost surely convergent as $t \to \infty$, and find the limit.

3. Let $X_1, X_2, \ldots$ be i.i.d. exponential with mean 1, and let $Z_1, Z_2, \ldots$ be i.i.d. standard normal $N(0, 1)$. Recall that as $t \to \infty$,

$$\mathbb{P}(Z_1 > t) \sim \frac{1}{\sqrt{2\pi}}\,\frac{1}{t}\,e^{-t^2/2},$$

where $\sim$ means the ratio converges to 1.

(a) Find a sequence $\{t_n\}$ so that for all $b \in \mathbb{R}$, as $n \to \infty$, $\mathbb{P}(\max_{i \leq n} X_i \leq t_n + b)$ has a limit $F(b) \in (0, 1)$.

(b) Find a sequence $s_n \to 0$ so that $\mathbb{P}(\max_{i \leq n} Z_i \leq \sqrt{2 \log n} + s_n)$ has a limit in $(0, 1)$.

(c) Show that $\mathbb{P}(\max_{i \leq n} X_i > \max_{i \leq n} Z_i) \to 1$ as $n \to \infty$.

MATH 507a **QUALIFYING EXAM** **August 2019**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems.

(1) Suppose $g : \mathbb{R} \to [0, \infty)$ is continuous, with $g > 0$ and $x^2/g(x) \to 0$ as $|x| \to \infty$. Suppose $X_n \implies X$ and $\sup_n E[g(X_n)] < \infty$. Show that $E[X_n^2] \to E[X^2]$. HINT: Truncate the function $h(x) = x^2$ at $\pm A$ to make a new function $h_A(x)$, for some A.

(2) Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $\mathcal{G} \subset \mathcal{F}$ a sub-σ-field. Suppose $E(X^2) < \infty$.

(a) Show that $E(XE(X \mid \mathcal{G})) = E(E(X \mid \mathcal{G})^2)$.

(b) Suppose $EX^2 = E(E(X \mid \mathcal{G})^2)$. Show that $X = E(X \mid \mathcal{G})$ a.s. HINT: $X = (X - E(X \mid \mathcal{G})) + E(X \mid \mathcal{G})$.

(3) Suppose $X_1, X_2, \dots$ are iid with distribution function $F(x) = 1 - e^{-2x^{1/2}}, x \geq 0$.

(a) Find $\alpha_n \nearrow \infty$ for which $\limsup_n X_n/\alpha_n = 1$ a.s.

(b) Let $M_n = \max(X_1, \dots, X_n)$. Find c such that $\limsup_n M_n/\alpha_n = c$ a.s. (Prove your answer.)

MATH 507a **QUALIFYING EXAM** **January 2020**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems.

(1) Let $\{X_n, n \geq 1\}$ be iid uniform in [0,1]. For which values of $\gamma > 0$ does the series $\sum_n (X_n + n^{-\gamma})^{n^{\gamma+1}}$ converge a.s.?

HINT: What happens when $X_n + n^{-\gamma} \geq 1$? Consider also events $X_n + n^{-\gamma} < 1 - \epsilon_n$ for some natural choice of ϵ_n.

(2) Let $\{X_i\}$ be iid nonnegative random variables with mean μ and variance $\sigma^2 \in (0, \infty)$. Let $S_n = \sum_{i=1}^n X_i$. Show that

$$\sqrt{S_n} - \sqrt{n\mu} \implies N\left(0, \frac{\sigma^2}{4\mu}\right) \quad \text{in distriubution, as } n \to \infty.$$

HINT: For $T_n = S_n - n\mu$ we have $S_n = n\mu\left(1 + \frac{T_n}{n\mu}\right)$. Also, you may assume Slutsky's Theorem: if $U_n \implies U$ in distribution and $V_n \to 0$ in probability, then $U_n + V_n \implies U$.

(3)(a) Suppose $\varphi(t)$ is a characteristic function of some random variable. Show that Re $\varphi(t)$ is also a characteristic function.

(b) Suppose X_1, X_2 are iid. Show that the distribution of $X_1 - X_2$ cannot be the uniform distribution over any interval $[a, b]$. HINT: This doesn't use part (a), but part (a) may get you thinking in a useful direction.

MATH 507a **QUALIFYING EXAM** **September 2020**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve.

(1) Let $X_1, X_2, \dots$ be i.i.d. nonnegative random variables with $P(X_1 > 0) > 0$.

(a) Show that $\limsup_n X_n^{1/n} \geq 1$ a.s. HINT: Consider events $\{X_n^{1/n} \geq 1 - \epsilon \text{ i.o.}\}$.

(b) Let $c > 1$. Show that $P(X_n^{1/n} \geq c \text{ i.o.}) = 1$ if and only if $E(\log^+ X_1) = \infty$. (Here $\log^+ x$ means $\log x$ if $\log x > 0$, and 0 otherwise.) HINT: $E(\log^+ X_1)$ is an integral, but it can be compared to a sum in a standard way. Also, for $t > 0$, the statement $\log^+ x > t$ is the same as $\log x > t$.

(c) Show that the only possible values of $\limsup_n X_n^{1/n}$ are 1 and ∞.

(2) Suppose $X_1, X_2, \dots$ are iid with characteristic function $\varphi(t) = 1 - \beta|t|^\alpha + o(|t|^\alpha)$ as $t \to 0$. Let Z have characteristic function $\psi(t) = e^{-|t|^\alpha}$ for some $\alpha \in (0, 2]$. Find b, θ (expressed in terms of α and β) so that S_n / bn^θ converges weakly to Z.

(3) Suppose X, Y are r.v.'s with $E(X^2) < \infty$, and let

$$\mathcal{H} = \{ \text{ all measurable } h : \mathbb{R} \to \mathbb{R} \text{ for which } E(h(Y)^2) < \infty \}.$$

(a) For $h \in \mathcal{H}$ show that $E\big[\big(X - E(X \mid Y)\big)\big(E(X \mid Y) - h(Y)\big)\big] = 0$.

(b) Show that the choice of h which minimizes $E[(X - h(Y))^2]$ over $h \in \mathcal{H}$ is $h(y) = E(X \mid Y = y)$, that is, $h(Y) = E(X \mid Y)$. HINT: Use (a).

MATH 507a **QUALIFYING EXAM** **January 2021**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve.

(1) For μ, ν probability measures on Borel sets in $\mathbb{R}$, define

$$\rho(\mu, \nu) = \inf\{\epsilon > 0 : \mu(A) \leq \nu(A^\epsilon) + \epsilon \text{ for all closed } A \subset \mathbb{R}\}.$$

Here $A^\epsilon = \{x \in \mathbb{R} : d(x, A) < \epsilon\}$, with d being Euclidean distance. Show that if μ_n, μ are probabillity measures satisfying $\rho(\mu_n, \mu) \to 0$ then $\mu_n \to \mu$ weakly.

(2) Let $X, X_1, X_2, \ldots$ be independent and identically distributed with characteristic function φ, and let $S_n = X_1 + \cdots + X_n$.

(a) Suppose $\varphi'(0) = 0$. Show that $S_n/n \to 0$ in probability. HINT: What other modes of convergence are sufficient to establish this?

(b) Suppose X has the symmetric density

$$f(x) = c\frac{1}{x^2 \log|x|}, \quad |x| \geq 4,$$

where c is the appropriate normalizing constant. Using part (a), show: $E(X)$ does not exist, but $S_n/n \to 0$ in probability. HINT: In representing $\varphi(t)$, integrate separately over $|x| \in [4, \epsilon/t]$ and $|x| \in (\epsilon/t, \infty)$, for some small ϵ. You should only need to bound the resulting integrals, not calculate them explicitly.

(3) Let $\beta \in (0,1)$ and let $X_1, X_2, \ldots$ be independent, with $X_i \sim \text{exponential}(i^{-\beta})$, that is, X_i has density $f(x) = i^{-\beta} e^{-i^{-\beta} x}$, $x \geq 0$. Let $M_n = \min_{i \leq n} X_i$. Show that

$$\sum_{n=1}^{\infty} M_n^\alpha$$

converges a.s. for all $\alpha > 1/(1-\beta)$. HINT: Consider events $\{M_n \geq n^{-\gamma}\}$ for general $\gamma > 0$.

MATH 507a **QUALIFYING EXAM** **September 2021**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve.

(1) Consider the following 3 possible properties of a random variable X with characteristic function φ:

(i) X is *discrete*, that is, there exists a countable $A \subset \mathbb{R}$ with $X \in A$ a.s.;

(ii) X is a *lattice random variable*, that is, there exist $a \in \mathbb{R}$ and $b \neq 0$ such that $X \in \{a + jb : j \in \mathbb{Z}\}$ with probability 1;

(iii) φ is periodic.

(a) Suppose $|\varphi(t)| = 1$ for some $t \neq 0$. Show that X is lattice. HINT: If $\varphi(t) = e^{i\theta}$ for some t, θ, what do you know about the random variable $e^{-i\theta}e^{itX}$?

(b) Prove, or disprove with a counterexample: if φ is periodic then X is discrete.

(c) Prove, or disprove with a counterexample: if X is discrete then φ is periodic.

(2) Fix $R > 0$ and let $B(0, R)$ be the ball of radius R in $\mathbb{R}^2$ centered at the origin. Let $X_0 = 1$, and suppose that for $n \geq 1$, given X_n, X_{n+1} is uniform in $B(0, R|X_n|)$, independent of $X_1, \ldots, X_n$. In other words, the random variables $X_{n+1}/|X_n|$ are independent, each uniform in $B(0, R)$.

(a) Find the density of $|X_1|$ and find $E(\log |X_1|)$.

(b) Show that there exists L_R such that $\frac{\log |X_n|}{n} \to L_R$ a.s. HINT: Relate $|X_n|$ to the ratios $X_{j+1}/|X_j|$.

(c) Find R_0 such that $X_n \to 0$ a.s. if $R < R_0$, but not if $R > R_0$.

(3)(a) Let $\{X_n, n \geq 1\}$ be random variables and $\mu \in \mathbb{R}$, let $Z_n = \sqrt{n}(X_n - \mu)$, and suppose $Z_n \to Z$ in distribution where Z is standard normal $N(0, 1)$. Let g be differentiable function with $g'(\mu) \neq 0$. Show that $\sqrt{n}(g(X_n) - g(\mu)) \to g'(\mu)Z$ in distribution. HINT: First show this for a.s. convergence.

(b) Let $\xi_1, \xi_2, \ldots$ be iid uniform in [0,1]. Find μ_n, σ_n such that

$$\frac{\left(\prod_{i=1}^n \xi_i\right)^{1/n} - \mu_n}{\sigma_n} \to Z \quad \text{in distribution,}$$

where Z is standard normal. HINT: Use (a).

SOME ANTIDERIVATIVES THAT MAY BE USEFUL IN THE EXAM (proof not required, and not all are necessarily needed):

$$\int \log t \, dt = t \log t - t + C, \quad \int (\log t)^2 \, dt = t(\log t)^2 - 2t \log t + 2t + C, \quad \int t \log t \, dt = \frac{t^2}{2} \log t - \frac{t^2}{4} + C.$$

Math 507a Qualifying Exam, Spring 2022

Answer all 3 questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway. Be aware of the passage of time, so that you can attempt all three problems. When a problem asks you to find something, you are expected to simplify the answer as much as possible.

Problem 1 : Let $E_1, E_2, \cdots$ be arbitrary events. Let $G := \limsup_n E_n$. Show that $\mathbb{P}(G) = 1$ if and only if $\sum_n \mathbb{P}(A \cap E_n) = \infty$ for all events A having $\mathbb{P}(A) > 0$.

Problem 2: Let $X_0, X_1, \cdots$ be i.i.d. and continuous. Let $N = \inf\{n \geq 1 : X_n > X_0\}$. Show that

$$\mathbb{P}(N > n) = \frac{1}{n+1},$$

and find $\mathbb{E}\, N$.

Problem 3: Let $X_1, \cdots$ be independent random variables such that $\sum_n n^{-2} \operatorname{Var}(\mathrm{X_n}) < \infty$. Prove that there is a random variable Y so that

$$\sum_{k=1}^{n} \frac{X_k - \mathbb{E}\, X_k}{k} \to Y$$

with probability 1. Do NOT assume that the X_i have the same distribution. Hint: Consider an appropriate S_n and use the Kolmogorov inequality for

$$\mathbb{P}(\sup_{k \geq n} |S_k - S_n| > \epsilon).$$

MATH 507a **QUALIFYING EXAM** **Fall 2022**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems.

(1) Let $\{x_j, j \geq 1\} \subset \mathbb{R}$, let $a_j > 0$ with $\sum_{j\geq 1} a_j = 1$, and let F be the distribution function with a jump of size a_j at each x_j. Suppose F_n has the same first n jumps, of size a_j at x_j for each $j \leq n$. (F_n may also have other jumps or may have a continuous part.) Show that $F_n \to F$ weakly.

(2) Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{G}, \mathcal{H}$ be sub–σ–algebras of $\mathcal{F}$ which are independent, that is, $P(A \cap B) = P(A)P(B)$ for all $A \in \mathcal{G}, B \in \mathcal{H}$. Let X be a random variable with $E|X| < \infty$. Find

$$E\Big(E(X \mid \mathcal{H}) \Bigm| \mathcal{G}\Big).$$

HINT: What is the integral of this function over a set in $\mathcal{G}$?

(3) Let $X_0, X_1, \ldots$ be iid with finite mean, not identically 0. Show that with probability 1, the infinite series

$$\sum_{n=0}^{\infty} X_n z^n$$

converges for all $z \in \mathbb{C}$ with $|z| < 1$, and diverges for all $z \in \mathbb{C}$ with $|z| \geq 1$.

HINT: Considering $z \in \mathbb{C}$ affects things very little—if we considered only $z \in \mathbb{R}$ the proof would be nearly all the same.

MATH 507a **QUALIFYING EXAM** **Spring 2023**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems.

(1) Let $X_1, X_2, \dots$ be independent (not necessarily iid) with positive finite variance, and $S_n = \sum_{k=1}^n X_k$. Let $v_n = \mathrm{Var}\,(\mathrm{S_n})$. Suppose that for some $q > 2$,

$$\lim_n v_n^{-q/2} \sum_{k=1}^n E(|X_k - EX_k|^q) = 0. \tag{1}$$

Show that $v_n^{-1/2}(S_n - ES_n)$ converges in distribution to a standard normal.

(2) Let $X_1, X_2, \dots$ be exponential r.v.'s with parameter λ (that is, density $\lambda e^{-\lambda x}, x \geq 0$.) Let $S_n = \sum_{k=1}^n kX_k$.

(a) Find the mean and variance of S_n.

(b) Show that there is a constant c such that $S_n/n^2 \to c$ in probability, and find c. HINT: $\sum_{k=1}^n k = n(n+1)/2$, $\sum_{k=1}^n k^2 = n(n+1)(2n+1)/6$.

(3) Let μ be a probability measure on $\mathbb{R}$ and φ its characteristic function.

(a) For $a \in \mathbb{R}$, express

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-ita} \varphi(t)\, dt$$

in terms of the measure μ.

(b) If $\lim_{|t|\to\infty} \varphi(t) = 0$, show that μ has no atoms (that is, no values x with $\mu(\{x\}) > 0$.)

MATH 507a **QUALIFYING EXAM** **January 2024**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems.

(1)(a) Suppose $X \leq 1$ a.s. If $EX = 1$, show that $X = 1$ a.s. (Write a proof, don't just cite a theorem.)

(b) If Y has characteristic function $\varphi(t)$ and $\varphi(t) = 1$ for some $t \neq 0$, show that Y is a discrete r.v., in fact show there exists $a > 0$ such that Y/a is integer-valued. HINT: Use (a).

(2) Let $X_1, X_2, \ldots$ be i.i.d. nonnegative random variables with $EX_1 < \infty$. Let $S_n = \sum_{i=1}^n X_i$.

(a) Show that $X_n/n \to 0$ a.s.

(b) Show that $\max(X_1, \ldots, X_n)/n \to 0$ a.s. HINT:

$$\max(X_1, \ldots, X_n) \leq \max(X_1, \ldots, X_k) + \max(X_{k+1}, \ldots, X_n),$$

for all $k \leq n$. Use (a).

(c) Show that

$$\max_{1 \leq k \leq n} \frac{X_k}{S_n + 1} \to 0 \quad a.s.$$

(3)(a) Suppose $0 \leq X_1 \leq X_2 \leq \ldots$ are random variables with $E(X_n^2) < \infty$, $EX_n \to \infty$, and $\text{var}(X_n)/(EX_n)^2 \to 0$. Show that $X_n \to \infty$ a.s.

(b) Let $\{A_n\}$ and $\{B_n\}$ be events such that

(i) the events B_n are mutually independent,

(ii) $P(A_n) \to 1$,

(iii) B_n is independent of A_n for each n,

(iv) $\sum_n P(B_n) = \infty$.

Show that $P(A_n \cap B_n \text{ i.o.}) = 1$. HINT: Instead of Borel–Cantelli, consider properties of $S_n = \sum_{k=1}^n 1_{A_k \cap B_k}$. Part (a) may be useful. Also, $P(A_j \cap B_j \cap A_k \cap B_k) \leq P(B_j \cap B_k)$.