MATH 505a **QUALIFYING EXAM** **January 2024**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems. When a problem asks you to find something, you are expected to simplify the answer as much as possible.

(1) Let X_1, X_2 be independent with Poisson(λ_1) and Poisson(λ_2) distribution, respectively.

(a) Find $P(X_1 = k \mid X_1 + X_2 = n)$ for $0 \leq k \leq n$. Simplify your answer so it does not involve a sum. Do the actual calculation, don't just cite a theorem.

(b) Find $E(X_1^2 + X_2^2 \mid X_1 + X_2 = n)$.

(2) Let X, Y be independent exponential random variables with parameters μ, λ respectively, that is, X and Y have densities $f_X(x) = \mu e^{-\mu x}$ on $f_Y(y) = \lambda e^{-\lambda x}$ for $x \geq 0$. Let $U = \max(X, Y)$ and $V = \min(X, Y)$.

(a) Find $E(U)$ and $E(V)$.

(b) Find the covariance $\text{cov}(U, V)$ in terms of λ and μ. HINT: This requires no integration.

(c) Find the density $f_Z(z)$ for $Z = V/U$.

(3) Fix $n \geq 2$ and let $X_1, \ldots, X_n$ be i.i.d. random variables uniform in $[0, 1]$. Let A denote the number of *ascents* in the sequence $(X_1, \ldots, X_n)$, that is, the number of indices $i \in \{1, \ldots, n-1\}$ with $X_i < X_{i+1}$. Let D denote the number of *descents*, that is, the number of indices $i \in \{1, \ldots, n-1\}$ with $X_i > X_{i+1}$.

(a) Find $P(A = 0)$ and find $E(A)$.

(b) Find $P(A = 1 \mid X_1 < X_2)$.

(c) Find $P(X_i < X_{i+1}, X_j > X_{j+1})$ for all (i, j).

(d) Find $\text{cov}(A, D)$.

Your answers should be functions of n. HINT: No integration is needed to do any part of this problem. Everything depends only on the ordering of the variables X_i.