MATH 505a **QUALIFYING EXAM** **January 31, 2012**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway.

1.) One hundred and one passengers bought tickets on a 101-seat carriage. One seat was reserved for each passenger. The first 100 passengers took the seats at random so that all 101! possible seating arrangements (with one empty seat) are equally likely. The last passenger insisted on taking the assigned seat. If that seat is occupied, then the passenger in that seat has to move to the corresponding assigned seat, and so on. Compute the expected value of the number M of passengers who have to change their seats. [HINTS: one method is to use a recursion in n, for n in the role of 101, for the expectation, without knowing the distribution of M. Another method is to find the distribution of M explicitly.]

2.) Suppose $\mathbb{P}(X = k) = p_k$ and $p_1 + p_2 + \cdots = 1$. Suppose that $X, X_1, X_2, \ldots, X_n$ are independent and identically distributed. Let $S = \sum_{1 \le i < j \le n} 1(X_i = X_j)$ be the number of matching (unordered) *pairs*, and let $T = \sum_{1 \le i < j < k \le n} 1(X_i = X_j = X_k)$ be the number of matching (unordered) *trios*. For $r = 1, 2, 3, \ldots,$, let $f_r = \sum p_i^r$, so that $f_1 = 1$.
a) Give a simple expression for $\mathbb{E}\, S$ in terms of n, f_2.
b) Give a simple expression for $\mathbb{E}\, T$ in terms of n, f_3.
c) Give a simple expression for $\operatorname{Var} \mathrm{S}$ in terms of n, f_2, f_3, f_4.

3.) Let X, X_1, X_2, X_3, X_4 be independent standard exponentially distributed random variables, so that $\mathbb{P}(X > x) = e^{-x}$ for $x > 0$. Write $S_n = X_1 + \cdots + X_n$. The goal is to show that the triple $(S_1/S_4, S_2/S_4, S_3/S_4)$ is distributed like the order statistics of three independent standard uniform (0,1) random variables.
a) Give a reason why the density of S_4 is $f(t) = t^3 e^{-t}/6$ for $t > 0$. You may either quote the density for the Gamma family in general, or you may argue about the time of the fourth arrival in a standard, rate 1 Poisson process, or you may carry out the four-fold convolution!
b) With (U_1, U_2, U_3) distributed uniformly in the cube $(0,1)^3$, and $U_{[i]}$ defined to be the ith smallest of U_1, U_2, U_3, show why the density of $(U_{[1]}, U_{[2]}, U_{[3]})$ is $g(x, y, z) = 6$, on the region $0 < x < y < z < 1$.
c) Show, with detail, why the triple $(S_1/S_4, S_2/S_4, S_3/S_4)$ is distributed like the order statistics of three independent standard uniform (0,1) random variables, AND that this triple is independent of S_4. This should include calculation of both a 4 by 4 Jacobian matrix, and calculation of its determinant, also known as the "Jacobian".
d) Consider three independent uniform $(0, 1)$ variables. Compute the probability that the largest exceeds the sum of the other two.

Math 505a 2012 Fall Qualifying Exam

1. In the Polya Urn model, $w \geq 1$ white balls and $b \geq 1$ black balls are placed in an urn at time 0, and at times $1, 2, \ldots$ a ball is chosen uniformly from the urn independent of the past, and replaced back into the urn with one additional ball of the same color.

a. A vector $(X_1, \ldots, X_n)$ of random variables is said to be exchangeable

$$(X_1, \ldots, X_n) =_d (X_{\pi(1)}, \ldots, X_{\pi(n)}) \quad \text{for all permutations } \pi$$

where $=_d$ denotes equality in distribution. If X_i is the indicator that a white ball is drawn from the urn at time i, prove that $(X_1, \ldots, X_n)$ is exchangeable.

b. Find the mean and variance of $S_n = X_1 + \cdots + X_n$, the total number of white balls added to the urn up to time n.

2. With a and b positive numbers, a needle of length $l \in (0, \min(a, b)]$ is dropped randomly on a rectangular grid consisting of an infinite number of parallel lines distance a apart, and, perpendicular to these, an infinite number of parallel lines distance b apart. Let A and B, respectively, be the events that the needle intersects the group of lines at distance a and b apart.

a. Show $P(A) = \frac{2l}{a\pi}$ and $P(B) = \frac{2l}{b\pi}$. Hint: The angle θ giving the orientation of the needle might be taken as uniform from $[0, 2\pi)$, but by symmetry, one may assume that the angle is uniformly taken from $[0, \pi/2)$.

b. Determine $P(A \cap B)$ and verify that A and B are strictly negatively correlated, that is, that $P(A \cap B) < P(A)P(B)$.

3. A total of k boys and $n-k$ girls sit around a circular table, with all $n!$ arrangements equally likely. Compute the mean and variance of the number Y of pairs of boy/girl neighbors. Note: In the circular arrangement GBGGBB, since the first G and last B are neighbors, $Y = 4$.

Math 505a 2013 Spring Qualifying Exam

1. a) Let X and Y be square integrable random variables such that

$$E(X|Y) = Y \quad \text{and} \quad E(Y|X) = X. \tag{1}$$

Show that

$$P(X = Y) = 1. \tag{2}$$

b) Prove that (1) implies (2) under the weakened assumption that X and Y are integrable.

2. Suppose k balls are tossed into n boxes, with all n^k possibilities equally likely. Let D be the number of boxes that contain exactly 2 balls.

a) Compute $p := P($ exactly 2 balls land in box 1$)$.

b) In terms of p, give an exact expression for the mean ED.

c) Compute $r := P($ exactly 2 balls land in box 1 and exactly 2 balls land in box 2$)$.

d) Give an exact expression for the second moment ED^2 in terms of p and r.

e) Compute the variance of D.

3. a) Suppose $g(u) := Eu^S$ is the probability generating function of a non-negative integer valued random variable S satisfying $P(S > 0) > 0$. Let T be distributed as S, conditional on the event $S > 0$. Express $h(u) := Eu^T$, the probability generating function of T, in terms of $g(u)$.

In parts b) and c) below, N is a nonnegative integer valued random variable with probability generating function $f(u) := Eu^N$, and S is the number of heads in N tosses of a $p \in (0, 1)$ coin, with all coin tosses having probability p of coming up heads, independently of each other and of N.

b) Write the probability generating function $g(u) := Eu^S$ of S in a simple form.

c) Now combine parts a) and b): what is the probability generating function h of the number T of heads, in N tosses of a p-coin, conditional on getting at least one head, when N has probability generating function f?
Parts d,e) can be worked on even if you are stumped by a,b,c).

d) Suppose someone claims that for $\alpha \in (0, 1)$, the function

$$f(u) := 1 - (1 - u)^\alpha$$

is a probability generating function of a nonnegative, non constant integer valued random variable N. What properties of f must you check? Is the hypothesis $\alpha > 0$ used? What happens in the cases $\alpha = 0, \alpha = 1$ and $\alpha > 1$?

e) Combine parts a)-d), that is suppose $\alpha \in (0, 1)$, N has the generating function $f(u) := 1 - (1 - u)^\alpha$, and T is the number of heads in N tosses of a p-coin, conditional on getting at least one head. Do N and T have the same distribution?

MATH 505a QUALIFYING EXAM September , 2013. One hour and 50 minutes

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway.

1.) In an infinite sequence of independent trials, events A, B are mutually exclusive, with $a = \mathbb{P}(A) > 0$ and $b = \mathbb{P}(B) > 0$.
a.) What is the probability that A will occur before B?
b.) In repeated independent tosses of a pair of fair dice, what is the probability that the sum 3 will occur before the sum 7?

2.) Let X and Y be independent, standard normal. Let $W = X + Y$ and $Z = X - Y$.
a.) Show that W and Z are independent.
b.) Simplify $\mathbb{E}\,(X + 2Y|Z)$.
c.) Simplify $\mathbb{E}\,(X|X > 0)$.

3.) n balls are placed into d boxes at random, with all d^n possibilities equally likely. Assume $d > 8$. Let X be the number of empty boxes.

a) Calculate and simplify: $\mathbb{E}\,X =$ _____

b) Calculate and simplify: $\mathrm{Var\,X} =$ _____

c) Let A be the event that boxes 1,2,3,4 are all empty, B be the event that boxes 3,4,5,6 are all empty, and C be the event that boxes 5,6,7,8 are all empty. Compute exactly, $\mathbb{P}(A \cup B \cup C) =$ _____

d) Let D be the event that no box receives more than 1 ball. Fix $a \in (0, 1)$. If both $n, d \to \infty$ together, what relation must they satisfy in order to have $\mathbb{P}(D) \to a$?

MATH 505a QUALIFYING EXAM February 6, 2014. One hour and 50 minutes, starting at 5pm.

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway.

1. A person plays a sequence of m games. He wins the nth game with probability a_n independently of the other games. Every time that he wins *two* consecutive games, he is rewarded with \$1; let R be the total reward, so that $0 \le R \le m-1$.

(i) As a function of $a_1, a_2, \ldots, a_m$, give exact expressions for $\mathbb{E}\, R$ and $\mathrm{Var}\,\mathrm{R}$.

(ii) Now suppose that $m = 100$ and $a_n = .1$ for all n. Simplify numerically $\mathbb{E}\, R$ and $\mathrm{Var}\,\mathrm{R}$.

2. Assume that $X_1, X_2, \cdots$ are independent and identically distributed, each with density

$$f(x) = x/2 \text{ for } 0 < x < 2.$$

For each of the following random variables, simplify the density *or* cumulative distribution function; you may choose either one, for each random variable.

a) $S = X_1 + X_2$.

b) $L = \min(X_1, \ldots, X_{100})$.

c) $R = X_1/X_2$.

d) $M =$ the 10^{th} smallest of $X_1, \ldots, X_{100}$.

3. Let $X_1, X_2, \cdots$ be uncorrelated random variables with $E[X_i] = \mu$ and $var(X_i) \le C < \infty$. If $S_n = X_1 + \cdots + X_n$, show that as $n \to \infty$, $S_n/n \to \mu$ in probability. That is, prove that for any $\varepsilon > 0$,

$$\lim_n \mathbb{P}\left(\left|\frac{S_n}{n} - \mu\right| \ge \varepsilon\right) \quad = 0.$$

Even if you are not comfortable with limits, simply give the best upper bound that you can, of the form

$$\mathbb{P}\left(\left|\frac{S_n}{n} - \mu\right| \ge \varepsilon\right) \le \underline{\qquad}.$$

MATH 505a QUALIFYING EXAM September 23, 2014. One hour and 50 minutes, starting at 2pm.

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway.

1. Let A and B be two events with $0 < \mathbb{P}(A) < 1$, $0 < \mathbb{P}(B) < 1$. Define the random variables $\xi = \xi(\omega)$ and $\eta = \eta(\omega)$ by

$$\xi(\omega) = \begin{cases} 5 & \text{if } \omega \in A; \\ -7 & \text{if } \omega \notin A; \end{cases} \qquad \eta(\omega) = \begin{cases} 2 & \text{if } \omega \in B; \\ 3 & \text{if } \omega \notin B. \end{cases}$$

True or false: the events A and B are independent if and only if the random variables ξ and η are uncorrelated? If you think this is true, then provide a proof. If you think this is false, then give a counter-example.

2. n people each roll one fair die. For each (unordered) pair of people that get the same number of spots, that number of spots is scored, with S for the total score achieved among the $\binom{n}{2}$ pairs of people. For example, if there are $n = 10$ people, and they roll 1,2,2,2,3,4,4,4,4,6 then $S = 2+2+2+4+4+4+4+4+4$ since there are three pairs of people matching 2 and six $= \binom{4}{2}$ pairs of people scoring 4.

(a) Simplify $\mathbb{E}S$.

(b) Simplify $\mathbb{E}S^2$.

[HINT: Consider S as the sum of $\binom{n}{2}$ random variables $S_{i,j}$, where $S_{i,j}$ is k if persons i and j both roll k, and zero otherwise.]

3. Let X be a standard normal random variable and, for $a > 0$, define the random variable Y_a by

$$Y_a = \begin{cases} X, & \text{if } |X| < a, \\ -X, & \text{if } |X| > a. \end{cases}$$

(a) Verify that Y_a is a standard normal random variable.

(b) Express $\rho(a) = \mathbb{E}(XY_a)$ in terms of the probability density function $\varphi = \varphi(x)$ of X.

(c) Is there a value of a for which $\rho(a) = 0$?

(d) Does the pair (X, Y_a) have a bivariate normal distribution? Explain your reasoning.

MATH 505a GRADUATE EXAM

FALL 2015

Answer as many questions as you can. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. If you cannot do part (a) of a problem, you can still get credit for (b), (c) etc. by assuming the answer to (a). Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1)(a) There are $n = 100$ balls, labelled 1 to n, and these are thrown into n boxes, also labelled 1 to n; all n^n outcomes are equally likely. Whenever ball i lands in box j, and $|i - j| \leq 1$, a point is scored, so the total score, call it X, can take on any value from 0 to n. Note: ball 1 in box n scores nothing, and ball n in box 1 scores nothing. Compute exactly, and simplify, as either an expression in n, or a decimal: $E(X)$ and $\mathrm{Var}(X)$.

(b) Pick the closest approximation: in part (a), $\mathrm{P}(X = 0)$ is close to 1, 1/3, 1/20, 1/100. You may reason informally, but you must describe your reasoning, which should involve the approximate distribution of X.

(c) Change the story in (a) to: there are $n = 100$ cards, numbered 1 to n, and they are placed in positions 1 to n around a circle at random, so that all $n!$ outcomes are equally likely. Whenver card i is placed in position i or $i + 1$, a point is scored, so the total score, call it Y, can take on any value from 0 to n. Here for card n, since the cards are in a circle, "position $n+1$" should be interpreted as position 1. Compute exactly, and simplify, as either an expression in n, or a decimal: $E(Y)$ and $\mathrm{Var}(Y)$.

(2)(a) A coin comes up heads with probability $p \in [0, 1]$; it is tossed independently n times, and X is the total number of heads. Simplify the generating function

$$G(s) = E(s^X),$$

and show how derivatives of G can be used to calculate $E(X)$ and $E(X^2)$.

(b) Random variables $U, U_1, U_2, \ldots, U_n$ are independent, and uniformly distributed in [0,1]. Let

$$Y = \sum_1^n 1_{\{U_i < U\}}$$

be the sum of indicators, counting how many of the U_i are less than U. Simplify the generating function

$$H(s) = E(s^Y),$$

and then simplify the ratio, $\mathrm{P}(Y = 2)/\mathrm{P}(Y = 1)$. HINT: conditionally on $U = p$, this Y is distributed as the X in part (a).

(3) Let $X_1, \ldots, X_n$, $Y_1, \ldots, Y_m$ be iid uniform on $(0, 1)$.

(a) Compute the probability density function of $\max(Y_1, \ldots, Y_m)$.

(b) Compute the probability density function of

$$\frac{\max(X_1, \ldots, X_n)}{\max(Y_1, \ldots, Y_m)}.$$

HINT: One method (not the only one) is to use conditioning.

MATH 505a GRADUATE EXAM

Fall 2016

Answer as many questions as you can. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. If you cannot do part (a) of a problem, you can still get credit for (b), (c) etc. by assuming the answer to (a). Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) You have a choice to roll a fair die either 100 times or 1000 times. For each of the following outcomes, state whether it is more likely with 100 rolls, or with 1000 rolls. Justify your answer, but you do not need to give a full formal proof.

(a) The number 1 shows on the die between 15% and 20% of the time.

(b) The number showing is at most 3, at least half the time.

(c) The number showing is 2 or 5, at least half the time.

(2) Let X and Y be independent exponential random variables with parameters λ and μ (that is, $E(X) = 1/\lambda$ and $E(Y) = 1/\mu$), and let $Z = \min(X, Y)$.

(a) Show that Z is independent of the event $X < Y$. In other words, show the event $Z \leq t$ is independent of $X < Y$ for all t.

(b) Find the distribution of $\max(X - Y, 0)$.

(3) Let $X_1, X_2, \ldots$ be iid with characteristic function φ. Let N be independent of the X_i's with $P(N = n) = 2^{-n}$ for all $n \geq 1$. Let $Y = \sum_{i=1}^{N} X_i$. Find the characteristic function of Y.

(4) $n \geq 4$ men, among whom are Alfred, Bill, Charles and David, stand in a row. Assume that all possible orderings of the n men are equally likely.

(a) Find the probability that Charles stands somewhere between Alfred and Bill. (Note this does not mean they are necessarily adjacent—there might be other people between Alfred and Bill.)

(b) Find the probability that David stands somewhere between Alfred and Bill given that Charles stands somewhere between Alfred and Bill.

(c) Find the expected value and variance of the number of men out of n who stand between Alfred and Bill. (Note Alfred and Bill themselves are not counted in this number.)

MATH 505a QUALIFYING EXAM Monday, February 9, 2015. One hour and 50 minutes, starting at 5pm.

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway.

1. Let $X_n,\ n \geq 1$, be independent random variables such that each X_n has Poisson distribution with mean λ_n. Prove that if $\sum_{n\geq 1} \lambda_n = +\infty$, then

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n X_k}{\sum_{k=1}^n \lambda_k} = 1$$

in probability.

2. A deck of cards is shuffled thoroughly. Someone goes through all 52 cards, scoring 1 each time 2 cards of the same value are consecutive. For example 9H,8H,7D,6C,7S,7H,7C, scores 2, once due to 7 of spades next to 7 of hearts, and once more 7 of hearts next to 7 of clubs. Write X for the total score.

a) Compute $\mathbb{E}X$.

b) Compute $\text{Var}X$.

c) Compute $\mathbb{P}(X = 39)$.

d) In the line below, circle the number that you think is the closest to the value $\mathbb{P}(X = 0)$ and briefly explain your choice.

$$\frac{1}{1000}, \quad \frac{1}{500}, \quad \frac{1}{100}, \quad \frac{1}{50}, \quad \frac{1}{20}, \quad \frac{1}{10}, \quad \frac{1}{5}, \quad \frac{1}{2}.$$

3. Let $S_0, S_1, S_2, \ldots$ be a simple symmetric random walk, i.e. $\mathbb{P}(S_i - S_{i-1} = 1) = \mathbb{P}(S_i - S_{i-1} = -1) = 1/2$, with independent increments. Let $T = \min\{n > 0 : S_n = 0\}$ be the hitting time to zero. Write $\mathbb{P}_a$ for probabilities for the walk starting with $S_0 = a$.

a) What does the reflection principle say about $\mathbb{P}_a(S_n = i, T \leq n)$, for $a > 0$, and $i, n \geq 0$?

b) What does the reflection principle say about $\mathbb{P}_a(S_n \geq i, T > n)$, for $a > 0$, and $i, n \geq 0$? [Hint: telescoping series]

c) For fixed $a > 0$, give asymptotics for $\mathbb{P}_a(T > n)$ as $n \to \infty$. [HINT: Stirling's formula is that $n! \sim \sqrt{2\pi n}\,(n/e)^n$.]

d) Simplify, for fixed $a > 0$,

$$\lim_{n\to\infty} \frac{\mathbb{P}_{a+1}(T > n)}{\mathbb{P}_a(T > n)}.$$

MATH 505a GRADUATE EXAM

Spring 2016

Answer as many questions as you can. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. If you cannot do part (a) of a problem, you can still get credit for (b), (c) etc. by assuming the answer to (a). Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) A stick of length 1 is broken at a point uniformly distributed over its length.

(a) Find the mean and variance of the sum S of the squares of the lengths of the two pieces.

(b) Find the density function of the product M of the lengths of the two pieces. Note that $M \in [0, \frac{1}{4}]$.

(2) There are two types of batteries in a bin. The life span of type i is an exponential random variable with mean $\mu_i, i = 1, 2$. The probability of type i battery to be chosen is p_i, with $p_1 + p_2 = 1$. Suppose a randomly chosen battery is still operating after t hours. What is the probability that it will still be operating after an additional s hours?

(3) Fix positive integers $m \leq n$ with $n > 4$. Suppose m people sit at a circular table with n seats, with all $\binom{n}{m}$ seatings equally likely. A seat is called *isolated* if it is occupied and both adjacent seats are vacant. Find the mean and variance of the number of isolated seats.

MATH 505a PROBABILITY GRADUATE EXAM
Spring 2017

Answer as many questions as you can. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Three points are chosen independently and uniformly inside the unit square in the plane. Find the expected area of the smallest closed rectangle that has sides parallel to the coordinate axes and that contains the three points. HINT: Consider what happens with just one coordinate.

(2) Suppose (X, Y) has joint density of the form $f(x, y) = g(\sqrt{x^2 + y^2})$ for $(x, y) \in \mathbb{R}^2$, for some function g. Show that $Z = Y/X$ has the Cauchy density $h(t) = 1/(\pi(1 + t^2)),\ t \in \mathbb{R}$. HINT: Polar coordinates.

(3) Assume $\sqrt{3} < C < 2$. Consider a sequence $X_1, X_2, X_3, \ldots$ of random variables where X_1 is uniform on $[0, 1]$, and where the conditional distribution of X_{n+1} given X_n is uniform on $[0, CX_n]$.

(a) Find the conditional expectation of $(X_{n+1})^r$ given X_n, for $r \geq 1$.

(b) Show that X_n converges to 0 in mean but not in mean square.

(c) Show that X_n converges to 0 almost surely.

(4) Suppose that n boys and m girls are arranged in a row, and assume that all possible orderings of the $n + m$ children are equally likely.

(a) Find the probability that all n boys appear in a single block.

(b) Find the probability that no two boys are next to each other.

(c) Find the expected number of boys who have a girl next to them on both sides.

MATH 505a **QUALIFYING EXAM** **August 2017.**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve.

When the problem asks you to compute something, you are expected to simplify the answer as much as possible (unless explicitly instructed otherwise).

1. Let X be uniform on $[1, 5]$, let Y be uniform on $[0, 1]$, and assume that X and Y are independent.

(a) Compute the probability density function of the product XY.

(b) Compute the cumulative distribution function of the ratio X/Y.

(c) Compute the characteristic function of the sum $X + Y$.

(d) Compute the moment generating function of the random variable $X - \ln(Y)$.

2. An urn contains $2n$ balls, coming in pairs: two balls are labeled "1", two balls are labeled "2",..., two balls are labeled "n". A sample of size n is taken without replacement. Denote by N the number of pairs in the sample. Compute the expected value and the variance of N. **You do not need to simplify the expression for the variance.**

3. Let $U_1, U_2, \ldots$ be iid random variables, uniformly distributed on $[0, 1]$, and let N be a Poisson random variable with mean value equal to one. Assume that N is independent of $U_1, U_2, \ldots$ and define

$$Y = \begin{cases} 0, & \text{if } N = 0, \\ \max\limits_{1 \le i \le N} U_i, & \text{if } N > 0. \end{cases}$$

Compute the expected value of Y.

MATH 505a **QUALIFYING EXAM** **January 2018.**

Answer all questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve.

When the problem asks you to compute something, you are expected to simplify the answer as much as possible.

1. Let X and Y be independent standard normal random variables and define $V = \min(X, Y)$. Compute the probability density function of V^2. The final answer should be an elementary function.

2. Consider positions 1 to n arranged in a circle, so that 2 comes after 1, 3 comes after 2, $\dots$, n comes after $n-1$, and 1 comes after n. Similarly, take 1 to n as values, with cyclic order, and consider all $n!$ ways to assign values to positions, bijectively, with all $n!$ possibilities equally likely. For $i = 1$ to n, let X_i be the indicator that position i and the one following are filled in with two consecutive values in increasing order, and define

$$S_n = \sum_{i=1}^{n} X_i, \quad T_n = \sum_{i=1}^{n} iX_i$$

For example, with $n = 6$ and the the circular arrangement 314562, we get $X_3 = 1$ since 45 are consecutive in increasing order, and similarly $X_4 = X_6 = 1$, so that $S_6 = 3, T_6 = 13$.

a) Compute the mean and the variance of S_n.

b) Compute the mean and the variance of T_n.

3. A box is filled with coins, each giving heads with some probability p. The value of p varies from one coin to another, and it is uniform in $[0, 1]$. A coin is selected at random; that one coin is tossed multiple times.

(a) Compute the probability that the first two tosses are both heads.

(b) Let X_n be the number of heads in the first n tosses. Compute $\mathbb{P}(X_n = k)$ for all $0 \le k \le n$.

(c) Let N be the number of tosses needed to get heads for the first time. Compute $\mathbb{P}(N = n)$ for all $n \ge 1$.

(d) Compute the expected value of N.

HINT: $\displaystyle\int_0^1 x^m (1-x)^\ell \, dx = \frac{m!\ \ell!}{(m+\ell+1)!}$ for nonnegative integers m, ℓ.

MATH 505a **QUALIFYING EXAM** **August 2018**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems. When a problem asks you to find something, you are expected to simplify the answer as much as possible.

1. Let X be an exponentially distributed random variable with $\mathbb{P}(X > t) = e^{-rt}$ for $t > 0$. Write X as the sum of its integer and fractional parts: $X = Y + Z$ with $Y = \lfloor X \rfloor \in \mathbb{Z}$ and $Z \in [0, 1)$.
(a) Find $\mathbb{E}\,X$
(b) Find $\mathbb{P}(Y = n)$, $n = 0, 1, 2, \ldots$.
(c) Find $\mathbb{E}\,Y$ and $\mathbb{E}\,Z$.
(d) Show that Y and Z are independent.

2. Let f and g be bounded nondecreasing functions on $\mathbb{R}$, and let X, Y be independent and identically distributed random variables.
(a) Show that
$$\mathbb{E}\left[\Big(f(X) - f(Y)\Big)\Big(g(X) - g(Y)\Big)\right] \geq 0.$$
(b) Show that $f(X)$ and $g(X)$ are positively correlated, that is,
$$\mathbb{E}\Big[f(X)g(X)\Big] \geq \mathbb{E}\left[f(X)\right] \cdot \mathbb{E}\left[g(X)\right].$$

3. Suppose that X and Y have joint density $f(x, y)$ given by $f(x, y) = ce^{-x}$ for $x > 0$ and $-x < y < x$ and $f(x, y) = 0$ otherwise.
(a) Show that $c = 1/2$.
(b) Find the marginal densities of X and Y, and the conditional density of Y given X.
(c) Find $\mathbb{P}(X > 2Y)$.

MATH 505a **QUALIFYING EXAM** **January 8, 2019**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems. When a problem asks you to find something, you are expected to simplify the answer as much as possible.

1. Suppose that each of 5 jobs is assigned at random to one of three servers A, B and C. [For example, one possible outcome would be that job 1 goes to server B, job 2 goes to server C, job 3 goes to server C, job 4 goes to server B and job 5 goes to server A. "At random" here means that there are 3^5 equally likely outcomes.]

(a) Find the probability that server C gets all 5 jobs.

(b) Let S be the number of servers that get exactly one job. Find $\mathbb{E}\, S$.

(c) Find the probability that no server gets more than 2 jobs.

(d) Take the same story, but with m in place of 5 for the number of jobs, and n in place of 3 for the number of servers. Find the variance of S, in terms of m and n.

2. (a) Suppose that X is Poisson with parameter λ. Find the characteristic function of X.

(b) Suppose that X_n is Poisson with parameter λ_n and that $\lambda_n \to \infty$ as $n \to \infty$. Show using characteristic functions that $(X_n - \lambda_n)/\sqrt{\lambda_n}$ converges in distribution, and describe the limiting distribution.

3. A stick of length 1 is broken into two pieces at a uniformly distributed random point.

(a) Find the expected length of the smaller piece.

(b) Find the expected value of the ratio of the smaller length over the larger.

(c) Suppose the larger piece is then broken at a random point, uniformly distributed over its length, independent of the first break point. There are then three pieces. Find the probability the longest of the three has length more than 1/2.

MATH 505a **QUALIFYING EXAM** **August 2019**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems.

(1) Suppose A, B, C are pairwise independent, $A \cap B \cap C = \emptyset$, and $P(A) = P(B) = P(C) = p$.

(a) What is the largest possible value of p?

(b) Is it possible that $P(A \cup B \cup C) = 1$? Prove or disprove.

(2) Consider two coins: coin 1 shows heads with probability p_1 and coin 2 shows heads with probability p_2. Each coin is tosses repeatedly. Let T_i be the time of first heads for coin i, and define the event $A = \{T_1 < T_2\}$.

(a) Find $P(A)$. HINT: One possible method is to condition on one of the variables.

(b) Find $P(T_1 = k \mid A)$ for all $k \geq 1$.

(3) Players A and B are having a table tennis match; the first player to win 3 games wins the match. One of the players is better than the other; this better player wins each game with probability 0.7. Carl comes to watch the match. He does not know who is the better player so (based on Carl's information) A, B each initially have probability 0.5 to be the better player. Then Carl sees A win 2 of the first 3 games.

(a) What is now the probability (after the 3 games, based on Carl's information) that A is the better player? Simplify your answer to a single fraction or decimal.

(b) What is now the probability (after the 3 games, based on Carl's information) that A will go on to win the match?

NOTE: Express your answer for (b) in terms of numbers; you do not need to simplify to a single number. An answer in a form like $\frac{5}{4} + 7(2 - \frac{9}{5})$ is OK.

(4) Suppose X_n is binomial with parameters (n, p) with $0 \leq p \leq 1$, and X is Poisson(λ).

(a) Find the moment generating function of X_n.

(b) Suppose $n \to \infty$ and $p = p_n \to 0$ with $np \to \lambda \in (0, \infty)$. Show that $P(X_n = k) \to P(X = k)$ as $n \to \infty$, for all $k \geq 0$. HINT: $(1 - \frac{c_n}{n})^n \to e^{-c}$ if $c_n \to c$.

(c) For n, p as in part (b), show that $P(X_n > k) \to P(X > k)$ as $n \to \infty$, for all $k \geq 0$.

MATH 505a **QUALIFYING EXAM** **January 2020**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems.

(1) Each pack of bubble gum contains one of n types of coupon, equally likely to be each of the types, independently from one pack to another. Let T_j be number of packs you must buy to obtain coupons of j different types. Note that $T_1 = 1$ always.

(a) Find the distribution and expected value of $T_2 - T_1$ and of $T_3 - T_2$.

(b) Compute $\mathbb{E}\, T_n$.

(c) Fix k and let A_i be the event that none of the first k packs you buy contain coupon i. Find $P(A_1 \cup A_2 \cup A_3 \cup A_4)$. Then fix $\alpha > 0$, take $k = \lfloor \alpha n \rfloor$ and find the limit of this probability as $n \to \infty$. Here $\lfloor x \rfloor$ denotes the largest integer $\leq x$. HINT: Consider probabilities $P(A_i), P(A_i \cap A_j)$, etc.

(d) Assume there are $n = 4$ coupon types; find $P(T_4 > k)$ for all $k \geq 4$. HINT: This is short if you use what you've already done.

(2) Let X be exponential(λ) (that is, density $f(x) = \lambda e^{-\lambda x}$.) The *integer part* of X is $\lfloor X \rfloor = \max\{k \in \mathbb{N} : k \leq X\}$. The *fractional part* of X is $X - \lfloor X \rfloor$. Show that $\lfloor X \rfloor$ and $X - \lfloor X \rfloor$ are independent.

HINT: In general, two random variables U, V are independent if the distribution of V conditioned on $U = u$ doesn't depend on u.

(3) Let X_1, X_2, X_3 be i.i.d. uniform in $[0, 1]$. Let $X_{(1)}$ be the smallest of the 3 values, $X_{(2)}$ the second smallest, and $X_{(3)}$ the largest.

(a) Find the distribution function and the expected value for $X_{(1)}$.

(b) Find the distribution function and the density of $X_{(2)}$.

MATH 505a **QUALIFYING EXAM** **September 2020**

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all four problems.

(1) Let X_n have binomial $B(n,p)$ distribution.

(a) Find $E\left(\frac{1}{X_n+1}\right)$. Simplify your answer so it does not involve a sum up to $n, n+1$, etc.

(b) Suppose $p = p_n$ and $np_n \to \lambda$ as $n \to \infty$, with $\lambda \in (0,\infty)$. Find $\lim_n E\left(\frac{1}{X_n+1}\right)$. Is it the same as $\lim_n \frac{1}{E(X_n+1)}$?

(2) Let X, Y be independent with $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$ distribution.

(a) Find $P(X = k \mid X + Y = n)$ for $0 \le k \le n$. Simplify your answer so it does not involve a sum. Do the actual calculation, don't just cite a theorem.

(b) Find $E(X^2 + Y^2 \mid X + Y = n)$.

(3) The county hospital is located at the center of a square whose sides are 2 miles wide. If an accident occurs within this square, then the hospital sends out an ambulance. The road network is rectangular, so the travel distance from the hospital, at coordinates $(0,0)$, to the point (x,y) is $|x| + |y|$. If an accident occurs at a point that is uniformly distributed in the square, find the mean and variance of the travel distance of the ambulance.

(4) Let X be a finite set X, and let P and Q be probabilities on X. Define the total variation distance between P and Q by

$$||P - Q||_{TV} = \frac{1}{2}\sum_{x\in X} |P(x) - Q(x)|.$$

Prove that

$$||P - Q||_{TV} = \max_{A\subseteq X} |P(A) - Q(A)|,$$

where the maximum is over subsets A of X.

MATH 505a **QUALIFYING EXAM** **January 2021**

Answer all 3 questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all four problems.

(1) A permutation π on n symbols is said to have i as a fixed point if $\pi(i) = i$.

a) Find the probability p_n that a random permutation of n symbols has no fixed points. HINT: Principle of inclusion and exclusion. (Your answer may involve a finite sum, which you don't need to simplify.)

b) Let S be a subset of $\{1, 2, ..., n\}$ of size k. Find the probability that the set of fixed points of a random permutation on n symbols is equal to S, and find the probability that a permutation has exactly k fixed points. HINT: If you didn't find the values p_j in part (a), you can still give answers for (b) expressed in terms of one or more p_j's.

c) Show that as n tends to infinity, the distribution of the number of fixed points converges to a Poisson(1) distribution.

(2) Let $\{S_n, n \geq 0\}$ be symmetric simple random walk, that is, $S_n = \sum_{i=1}^n \xi_i$ with $\xi_1, \xi_2, \ldots$ i.i.d. satisfying $P(\xi_1 = 1) = P(\xi_1 = -1) = 1/2$. Let $T = \min\{n : S_n = 0\}$, and write P_a for probabilities when the walk starts at $S_0 = a$. By *basic probabilities for* $\{S_n\}$ we mean probabilities of the form $P_0(S_n = k), P_0(S_n \geq k)$, or $P_0(S_n \leq k)$, all of which corresponding to starting at $S_0 = 0$..

(a) For $a \geq 1, i \geq 1, n \geq 1$, express $P_a(S_n = i, T \leq n)$ and $P_a(S_n = i, T > n)$ in terms of finitely many basic probabilities. HINT: Reflection principle.

(b) For $a \geq 1, i \geq 1, n \geq 1$, show that

$$P_a(T > n) = \sum_{j=1-a}^{a} P_0(S_n = j).$$

HINT: Use (a) and look for cancellation.

(c) You may take as given that $P_0(S_{2m} = 2j) \sim 1/\sqrt{\pi m}$ as $m \to \infty$ for each fixed $j \in \mathbb{Z}$; here $\sim$ means the ratio converges to 1. Use this to find c, α such that $P_a(T > n) \sim c/n^\alpha$ as $n \to \infty$, where $a > 0$. Does c or α depend on a? HINT: It's enough to consider even n—why?

(3) Let X, Y be independent standard normal $N(0, 1)$ random variables.

(a) Find a for which $U = X + 2Y$ and $V = aX + Y$ are independent.

(b) Find $E(XY \mid X + 2Y = a)$ for all $a \in \mathbb{R}$. HINT: Use (a).

MATH 505a **QUALIFYING EXAM** **September 2021**

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(1)(a) Let X be a non-negative random variable with finite expectation. Show that

$$\sum_{i=1}^{\infty} P(X \geq i) \leq E[X] < 1 + \sum_{i=1}^{\infty} P(X \geq i).$$

(b) Show that if X takes values only in $\{0, 1, \ldots, n\}$ for some n, then the first inequality in (a) in an equality:

$$\sum_{i=1}^{\infty} P(X \geq i) = E[X].$$

(c) Let M be the minimum value seen in 4 die rolls. Find $E[M]$. You don't need to simplify to one number, just get an expression in terms of numbers only.

(2) Suppose X and Y are independent continuous random variables with uniform distribution on $[0, 1]$.

(a) Find the density function of $X + 2Y$.

(b) Find the joint density function for $X - Y, X + Y$.

(3) Consider Bernoulli trials with success probability $p \in (0, 1)$. Let p_n be the probability of an odd number of successes in n trials.

(a) Express p_n in terms of p_{n-1}.

(b) Based on (a), for what value λ does $p_{n-1} = \lambda$ imply $p_n = \lambda$?

(c) Show that $\lim_n p_n = \lambda$, the value you found in (b). HINT: Write p_n as $\lambda + \epsilon_n$, for the λ you found in (b).

Math 505a Qual, Spring 2022

Instructions: Answer all 3 questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway. Be aware of the passage of time, so that you can attempt all three problems. When a problem asks you to find something, you are expected to simplify the answer as much as possible.

Problem 1:

a) Let X_1, X_2, X_3 be independent exponential random variables with parameter $\lambda = 1$. So $P(X_i > x) = e^{-x}$, $x > 0$. Find

$$E\left(\frac{X_1}{X_1 + X_2 + X_3}\right).$$

b) Let (X, Y) be independent uniforms on $[0, 1]$. Find the joint density function of X and $V = X + Y$. Find $f(x|v)$, the density function of X conditional on $V = v$. Also, find $E(X|V)$.

Problem 2: In an election, candidate A receives n votes, and candidate B receives m votes, where $n > m$. Assuming that all $\binom{m+n}{m}$ orderings are equally likely, show that the probability that A is always ahead in the count of votes is $(n - m)/(n + m)$.

Problem 3: Let n be a positive integer with prime factorization $n = p_1^{m_1} \cdots p_k^{m_k}$ for distinct primes $p_1, \cdots, p_k$ with $m_1, \cdots, m_k > 0$. Choose an integer N uniformly at random from the set $\{1, 2, \cdots, n\}$. Show that the probability that N shares no common prime factor with n is equal to

$$(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \cdots (1 - \frac{1}{p_k}).$$

(Hint: use inclusion-exclusion)

MATH 505a **QUALIFYING EXAM** **Fall 2022**

Answer all 3 questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems.

(1)(a) A standard deck of 52 cards has 13 cards of each suit (spades, hearts, diamonds, clubs.) A bridge hand consists of 13 randomly chosen cards from the deck (chosen without replacement.) The bridge hand is said to be *void* in a suit if it contains no cards of that suit. Find the probability that a bridge hand is void in at least one suit.

(b) Let A be an 10-element set and let B be a 4-element set, and let Φ be a random map from A to B. Show that

$$P(\Phi \text{ is not a surjection}) \leq 4 \cdot \left(\frac{3}{4}\right)^{10}.$$

Here a "random map" means one chosen uniformly among all possible maps.

(c) Find an exact formula for the probability in (b).

Your answers to (a) and (c) may involve factorials and/or quantities $\binom{n}{k}$; you need not simplify these any further.

(2) Let $n \geq 5$ and let $X_1, \ldots, X_n$ be iid, each taking integer values 0 through 9 with probability 1/10 each. Let N be the number of indices i for which $X_i = X_{i+1} = X_{i+2}$. Find the mean and variance of N.

Note there are no variables $X_{n+1}, X_{n+2}, \ldots$ defined, just X_1 through X_n. Also, N counts strings of 3 identical digits; if you get a string of more than 3 identical digits, there will be overlapping strings of 3 digits. So for example ..744442.., contributes 2 to the value of N, because 444 appears twice.

(3) Let $U_1, \ldots, U_n$ be iid uniform in [0,1], and let X_n be the second smallest of the values $U_1, \ldots, U_n$.

(a) Find $P(X_n > t)$ for $t \in [0, 1]$.

(b) Find c_n so $c_n - \log X_n$ converges in distribution, and find the distribution function of the limit. HINT: Given $a \in \mathbb{R}$, what is $\lim_n (1 + \frac{a}{n})^n$?

MATH 505a **QUALIFYING EXAM** **Spring 2023**

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(1) Suppose that each of k jobs is assigned at random to one of four servers A, B, C, and D. "At random" here means that there are 4^k equally likely outcomes.

Find the probability of the event E = (every server gets at least one job).

(2) Let X_n have a Poisson distribution with parameter n. Find constants a_n such that $\sqrt{X_n} - a_n$ converges in distribution, and find the limiting distribution.

HINT: What is the limit in distribution of $(X_n - n)/\sqrt{n}$?

(3) Let U, V be independent $N(0, 1)$ r.v.'s.

(a) Given $\mu_X, \mu_Y \in \mathbb{R}$, $\sigma_X, \sigma_Y > 0$ and $\rho \in [-1, 1]$, find a, b, c such that letting

$$(*) \qquad X = \mu_X + aU, \qquad Y = \mu_Y + bU + cV,$$

(X, Y) is multivariate normal with covariance matrix

$$\begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}.$$

(b) Find the conditional density of Y given $X = x$, expressed in terms of $\mu_X, \mu_Y, \sigma_X, \sigma_Y$, and ρ. HINT: Use $(*)$ directly.

(c) Let $\xi_1, \xi_2, \ldots$ be iid normal $N(0, 1)$ random variables and $S_n = \sum_{k=1}^{n} \xi_i$. For $m < n$, find the conditional density of S_m given $S_n = c$, for all c.

MATH 505a **QUALIFYING EXAM** **January 2024**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems. When a problem asks you to find something, you are expected to simplify the answer as much as possible.

(1) Let X_1, X_2 be independent with Poisson(λ_1) and Poisson(λ_2) distribution, respectively.

(a) Find $P(X_1 = k \mid X_1 + X_2 = n)$ for $0 \leq k \leq n$. Simplify your answer so it does not involve a sum. Do the actual calculation, don't just cite a theorem.

(b) Find $E(X_1^2 + X_2^2 \mid X_1 + X_2 = n)$.

(2) Let X, Y be independent exponential random variables with parameters μ, λ respectively, that is, X and Y have densities $f_X(x) = \mu e^{-\mu x}$ on $f_Y(y) = \lambda e^{-\lambda x}$ for $x \geq 0$. Let $U = \max(X, Y)$ and $V = \min(X, Y)$.

(a) Find $E(U)$ and $E(V)$.

(b) Find the covariance $\text{cov}(U, V)$ in terms of λ and μ. HINT: This requires no integration.

(c) Find the density $f_Z(z)$ for $Z = V/U$.

(3) Fix $n \geq 2$ and let $X_1, \ldots, X_n$ be i.i.d. random variables uniform in $[0, 1]$. Let A denote the number of *ascents* in the sequence $(X_1, \ldots, X_n)$, that is, the number of indices $i \in \{1, \ldots, n-1\}$ with $X_i < X_{i+1}$. Let D denote the number of *descents*, that is, the number of indices $i \in \{1, \ldots, n-1\}$ with $X_i > X_{i+1}$.

(a) Find $P(A = 0)$ and find $E(A)$.

(b) Find $P(A = 1 \mid X_1 < X_2)$.

(c) Find $P(X_i < X_{i+1}, X_j > X_{j+1})$ for all (i, j).

(d) Find $\text{cov}(A, D)$.

Your answers should be functions of n. HINT: No integration is needed to do any part of this problem. Everything depends only on the ordering of the variables X_i.