Numerical Analysis Preliminary Examination Spring 2024

December 14, 2023

Problem 1. Suppose $A \in \mathbb{R}^{n\times n}$ is symmetric and positive definite.

(a) Show that all the diagonal entries in A are positive and that the entry in A with largest absolute value lies on the diagonal of A.

Suppose we perform Gaussian elimination on the matrix A. Let $A^{(0)} = A$ and let $A^{(k)}$ be the $(n-k)\times(n-k)$ matrix in the lower right corner after k rounds.

(b) Show that each $A^{(k)}$ is symmetric positive definite.

Hint: Write $A = A^{(0)}$ *as*

$$A = \begin{bmatrix} \alpha & v^T \\ v & B \end{bmatrix}$$

and show that

$$A^{(1)} = B - \frac{vv^T}{\alpha}$$

and that for every $x = [x_2, x_3, \ldots, x_n]^T \in \mathbb{R}^{n-1}$*, if* $y = [y_1, x_2, \ldots, x_n]^T$ *and* y_1 *is suitably chosen , then* $x^T A^{(1)} x = y^T A y$*.*

(c) What is the significance of (b) to the execution of Gaussian elimination on the matrix A?

(d) Consider the norm on $n \times n$ matrices defined by $\|C\| = \max_{i,j} |C_{ij}|$ and let U be the upper triangular matrix we obtain from doing Gaussian elimination on A. Show

$$\|A^{(k)}\| \leq \|A\|$$

for $k = 1, 2, \ldots, n-1$ and that $\|U\| \leq \|A\|$.

(e) What is the significance of (d) to the execution of Gaussian elimination on the matrix A?

Problem 2. The following questions are related to eigenvalues and eigenvectors of a non-defective $n \times n$ matrix A.

(a) Let $A = (a_{ij})$ be such a matrix and let $r_i = \sum_{j \neq i} |a_{ij}|, i = 1, \ldots, n$. Show that for each eigenvalue λ of A at least one of the following inequalities hold:

$$|\lambda - a_{ij}| \leq r_i \qquad i = 1, \ldots, n$$

(b) Apply the results in part a) to the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

(c) Let λ be an eigenvalue of A. Show that for any induced norm of A we have: $|\lambda| \leq \|A\|$.

(d) Let $|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|$. That is λ_1 is the dominant eigenvalue of A. Show that for appropriate starting vectors x_0, the iteration $x_k = A^k x_0$ can be used to approximate λ_1 and its corresponding eigenvector. Furthermore, show that the rate of convergence is determined by the ratio $|\frac{\lambda_2}{\lambda_1}|$.

(e) Apply the method in part (d) to the matrix A in part (b). Do only two iterations.

Problem 3. Consider the stationary vector-matrix iteration given by

$$x_{k+1} = Mx_k + c \tag{1}$$

where $M \in \mathbb{C}^{n \times n}$, $c \in \mathbb{C}^n$, and $x_0 \in \mathbb{C}^n$ are given.

(a) If $x^* \in \mathbb{C}^n$ is a fixed point of (1) and $\|M\| < 1$ where $\| \cdot \|$ is any compatible matrix norm induced by a vector norm, show that x^* is unique and that $\lim_{k \to \infty} x_k = x^*$ for any $x_0 \in \mathbb{C}^n$.

(b) Let $\rho(M)$ denote the spectral radius of the matrix M and use the fact that $\rho(M) = \inf \|M\|$, where the infimum is taken over all compatible matrix norms induced by vector norms, to show that $\lim_{k \to \infty} x_k = x^*$ for any $x_0 \in \mathbb{C}^n$ if and only if $\rho(M) < 1$.

(c) Now consider the linear system

$$Ax = b \tag{2}$$

where $A \in \mathbb{C}^{n \times n}$ is nonsingular and $b \in \mathbb{C}^n$ are given. What are the matrix $M \in \mathbb{C}^{n \times n}$ and the vector $c \in \mathbb{C}^n$ in (1) in the case of the Jacobi iteration for solving the linear system given in (2)?

(d) Use Part (a) to show that if the matrix $A \in \mathbb{C}^{n \times n}$ is row diagonally dominant then the Jacobi iteration will converge to the solution of the linear system given in (2).

(e) Use Part (b) together with the Gershgorin Circle Theorem to show that if the matrix $A \in \mathbb{C}^{n \times n}$ is row diagonally dominant then the Jacobi iteration will converge to the solution of the linear system given in (2).