Preliminary/Qualifying Exam in Numerical Analysis (Math 502a) Spring 2012

Instructions

The exam consists of four problems, each having multiple parts. You should attempt to solve all four problems.

1. **Linear systems**

(a) What is the LU-decomposition of an n by n matrix A, and how is it related to Gaussian elimination? Does it always exist? If not, give sufficient condition for its existence.

(b) What is the relation of Cholesky factorization to Gaussian elimination? Give an example of a symmetric matrix for which Cholesky factorization does not exist.

(c) Let $\mathrm{C} = \mathrm{A} + \mathrm{iB}$ where A and B are real n by n matrices. Give necessary and sufficient conditions on A and B for C to be Hermitian, and give a nontrivial example of a 3 by 3 Hermitian matrix.

2. **Least squares**

(a) Give a simple example which shows that loss of information can occur in forming the normal equations. Discuss how the accuracy can be improved by using iterative improvement.

(b) Compute the pseudoinverse, $x^\dagger$, of a nonzero row or column vector, x, of length n. Let a = [1, 0] and let b = $[1, 1]^T$. Show that $(ab)^\dagger \neq b^\dagger a^\dagger$.

3. **Iterative Methods**

Consider the stationary vector-matrix iteration given by

$$x_{k+1} = Mx_k + c \qquad (1)$$

where $M \in C^{n\times n}$, $c \in C^n$, and , $x_0 \in C^n$ are given.

(a) If $x^* \in C^n$ is a fixed point of (1) and $\|M\| < 1$ where $\|\cdot\|$ is any compatible matrix norm induced by a vector norm, show that x^* is unique and that $\lim_{k\to\infty} x_k = x^*$ for any $x_0 \in C^n$.

(b) Let $r(M)$ denote the spectral radius of the matrix M and use the fact that $r(M) = inf\|M\|$, where the infimum is taken over all compatible matrix norms induced by vector norms, to show that $\lim_{k\to\infty} x_k = x^*$ for any $x_0 \in C^n$ if and only if $r(M) < 1$.

Now consider the linear system

$$Ax = b \qquad (2)$$

where $A \in C^{n\times n}$ nonsingular and $b \in C^n$ are given.

(c) What are the matrix $M \in C^{n\times n}$ and the vector $c \in C^n$ in (1) in the case of the Jacobi iteration for solving the linear system given in (2).

(d) Use part (a) to show that if $A \in C^{n\times n}$ is strictly diagonally dominant then the Jacobi iteration will converge to the solution of the linear system (2).

(e) Use part (b) together with the Gershgorin Circle Theorem to show that if $A \in C^{n\times n}$ is strictly diagonally dominant then the Jacobi iteration will converge to the solution of the linear system (2).

4. Computation of Eigenvalues and Eigenvectors

Consider an $n \times n$ Hermitian matrix A and a unit vector q_1. For $k = 2, \cdots n$, let $p_k = Aq_{k-1}$ and set

$$q_k = \frac{h_k}{\|h_k\|_2}, \qquad h_k = p_k - \sum_{j=1}^{k-1} \left(q_j^H \cdot p_k\right) q_j.$$

where $\|\cdot\|_2$ is the Euclidian norm in C^n.

(a) Show that the vectors q_k, for $k = 1, \cdots n,$ form an orthogonal set if none of the vectors h_k is the zero vector.

(b) Consider the matrix $Q^H A Q$. Use part (a) to show that it is a tridiagonal matrix (Hint: $[Q^H A Q]_{i,j} = q_i^H A q_j$).

(c) Suggest a possible approach that uses the result of part (b) to reduce the number of operations in the QR-algorithm for the computation of the eigenvalues of the matrix *A*.

Numerical Analysis Screening Exam, Fall 2012

Direct Methods for Linear Equations.

a. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite (SPD) matrix. There exists a nonsingular lower triangle matrix L satisfying $A = L \cdot L^t$. Is this factorization unique? If not, propose a condition on L to make the factorization unique.

b. Compute the above factorization for

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 13 & 8 \\ 1 & 8 & 14 \end{pmatrix}.$$

Iterative Methods for Linear Equations.

Consider the iterative method:

$$Nx_{k+1} = Px_k + b, k = 0, 1, \cdots,$$

where N, P are $n \times n$ matrices with $detN \neq 0$; and x_0, b are arbitraray $n-$dim vectors. Then the above iterates satisfy the system of equations

$$x_{k+1} = Mx_k + N^{-1}b, k = 0, 1, \cdots \tag{1}$$

where $M = N^{-1}P$. Now define $N_\alpha = (1 + \alpha)N, P_\alpha = P + \alpha N$ for some real $\alpha \neq -1$ and consider the related iterative method

$$x_{k+1} = M_\alpha x_k + N_\alpha^{-1} b, \quad k = 0, 1, \cdots, \tag{2}$$

where $M_\alpha = N_\alpha^{-1} P_\alpha$.

a. Let the eigenvalues of M be denoted by: $\lambda_1, \lambda_2, \cdots, \lambda_n$. Show that the eigenvalues $\mu_{\alpha,k}$ of M_α are given by:

$$\mu_{\alpha,k} = \frac{\lambda_k + \alpha}{1 + \alpha}, \quad k = 1, 2, \cdots, n$$

b. Assume the eigenvalues of M are real and satisfy: $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n < 1$. Show that the iterations in eq. (2) converge as $k \to \infty$ for any α such that $\alpha > \frac{1 + \lambda_1}{2} > -1$.

Eigenvalue Problem.

a. Let λ be an eigenvalue of a $n \times n$ matrix A. Show that $f(\lambda)$ is an eigenvalue of $f(A)$ for any polynomial $f(x) = \sum_{k=0}^{n} a_k x^k$.

b. Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a symmetric matrix satisfying:

$$a_{1i} \neq 0, \qquad \sum_{j=1}^{n} a_{ij} = 0, \qquad a_{ii} = \sum_{j\neq i} |a_{ij}|, \qquad i = 1, \cdots, n$$

Show all eigenvalues of A are non-negative and determine the dimension of eigenspace corresponding to the smallest eigenvalue of A.

Least Square Problem.

a. Let A be an $m \times n$ real matrix with the following singular value decomposition: $A = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T & V_2^T \end{pmatrix}^T$, where $U = (U_1\ U_2)$ and $V = (V_1\ V_2)$ are orthogonal matrices, U_1 and V_1 have $r = rank(A)$ columns, and Σ is invertible.

For any vector $b \in \mathbb{R}^n$, show that the minimum norm, least squares problem:

$$\min_{x \in S} \|x\|_2, \qquad S = \{x \in \mathbb{R}^n \mid \|Ax - b\|_2 = min\}$$

always has a unique solution, which can be written as $x = V_1 \Sigma^{-1} U_1^T b$.

b. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Using part a) above, find the minimum norm, least squares solution to the problem:

$$\min_{x \in S} \|x\|_2, \qquad S = \{x \in \mathbb{R}^n \mid \|Ax - b\|_2 = min\}$$

Hint: You can assume that the U in the SVD of A must be of the form $U = \begin{pmatrix} a & a \\ a & -a \end{pmatrix}$ for some real $a > 0$.

Numerical Analysis Screening Exam, Spring 2013

Problem 1. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite (SPD) matrix. At the end of the first step of Gaussian Elimination without partial pivoting, we have:

$$A_1 = \left(\begin{array}{c|ccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline 0 & & & \\ \vdots & & \hat{A} & \\ 0 & & & \end{array}\right)$$

a. Show that $\hat{A}$ is also a SPD.

b. Use the first conclusion to show the existence of the LU factorization and Cholesky factorization of any SPD.

Problem 2.

A matrix A with all non-zero diagonal elements can be written as $A = D_A(I-L-U)$ where D_A is a diagonal matrix with identical diagonal as A and matrices L and U are lower and upper triangular matrices with zero diagonal elements. The matrix A is said to be consistently ordered if the eigenvalues of matrix $\rho L+\rho^{-1}U$ are independent of $\rho \neq 0$. Consider a tri-diagonal matrix A of the form

$$A = \begin{pmatrix} \alpha & \beta & 0 & \cdots & 0 \\ \beta & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \beta \\ 0 & \cdots & 0 & \beta & \alpha \end{pmatrix}$$

with $|\alpha| \geq 2\beta > 0$.

a. Show that the matrix A is consistently ordered.

b. Show that if $\lambda \neq 0$ is an eigenvalue of the iteration matrix B_ω of the Successive Over Relaxation (SOR) method for matrix A

$$B_\omega = (I - \omega L)^{-1}((1-\omega)I + \omega U),$$

then $\mu = (\lambda + \omega - 1)(\omega\sqrt{\lambda})^{-1}$ is an eigenvalue of $L + U$.

Problem 3.

a. Assume that $v_1 = (1,1,1)^T$ is an eigenvector of a 3×3 matrix B. Find a real unitary matrix V such that the first column of the matrix $V^T BV$ contains all zeros except on the first row.

b. Consider a matrix A defined by

$$A = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 1 \\ -2 & 0 & 3 \end{pmatrix}$$

verify that $v_1 = (1,1,1)^T$ is an eigenvector of A and the first column of the matrix $V^T AV$ contains all zeros except on the first row where V is the matrix you obtained in (a).

c. Assume that $V^T AV$ has the form

$$V^T AV = \begin{pmatrix} * & * & * \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$$

Find a Schur decomposition of the matrix A. That is, find a unitary matrix U such that $U^H AU = R$ where R is an upper triangular matrix and U^H is the conjugate transpose of U.

Problem 4.

Consider a n by m matrix A and a vector $b \in \mathbb{R}^n$. A minimum norm solution of the least squares problem is a vector $x \in \mathbb{R}^m$ with minimum Euclidian norm that minimizes $\|Ax - b\|$. Consider a vector x^* such that $\|Ax^* - b\| \leq \|Ax - b\|$ for all $x \in \mathbb{R}^n$. Show that x^* is a minimum norm solution if and only if x^* is in the range of A^*.

Preliminary Exam in Numerical Analysis Fall 2013

Instructions

The exam consists of four problems, each having multiple parts. You should attempt to solve all four problems.

1. **Linear systems**
   For $x \in R^n$ and $M \in R^{n\times n}$ let $\|x\|$ denote the norm of x, and let $\|M\|$ denote the corresponding induced matrix norm of M. Let $S \in R^{n\times n}$ be nonsingular and define a new norm on R^n by $\|x\|_S = \|Sx\|$.

(a) Show that $\| \quad \|_S$ is in fact a norm on R^n.
(b) Show that $\| \quad \|_S$ and $\| \quad \|$ are equivalent norms on R^n.
(c) Show that the induced norm of $M \in R^{n\times n}$ with respect to the $\| \quad \|_S$ norm is given by $\|M\|_S = \|SMS^{-1}\|$.
(d) Let $\kappa(M)$ denote the condition number of $M \in R^{n\times n}$ with respect to the $\| \quad \|$ norm, let $\kappa_S(M)$ denote the condition number of $M \in R^{n\times n}$ with respect to the $\| \quad \|_S$ norm and show that $\kappa_S(M) \le \kappa(S)^2\kappa(M)$.

2. **Least squares**
   (a) Assume you observe four (x, y)data points: (0,1), (1, 1), (-1, -1), (2,0). You want to fit a parabola of the form $y = a + bx^2$ to these data points that is best in the least squares sense. Derive the normal equations for this problem and put them in matrix vector form (you do not need to solve the equations).
   (b) Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ and consider the linear system $Ax = b$, for $b \in R^3$. Find the QR or SVD decomposition of A and the rank of A.
   (c) For a given $b \in R^3$, state the condition such that the equation in part (b) has a solution, and the condition such that the solution is unique.
   (d) Find the pseudoinverse of the matrix A given in part (b).
   (e) For $b = \begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix}$ find the solution x to the system given in part (b).

3. **Iterative Methods**
   Consider the stationary vector-matrix iteration given by
   $$x_{k+1} = Mx_k + c \qquad (1)$$
   where $M \in C^{n\times n}$, $c \in C^n$, and , $x_0 \in C^n$ are given.

(a) Let $r(M)$ denote the spectral radius of the matrix M and show that if $\lim_{k\to\infty} x_k = x^*$ for any $x_0 \in C^n$, then $r(M) < 1$.

Now consider the linear system

$$Ax = b \tag{2}$$

where $A \in C^{n\times n}$ nonsingular and $b \in C^n$ are given.

(b) Derive the matrix $M \in C^{n\times n}$ and the vector $c \in C^n$ in (1) in the case of the Gauss-Seidel iteration for solving the linear system given in (2).

(c) Derive the matrix $M \in C^{n\times n}$ and the vector $c \in C^n$ in (1) in the case of the Successive Over Relaxation Method (SOR) with parameter θ for solving the linear system given in (2). (Hint: Use your answer in part (b) and write D as $D = \frac{1}{\theta}D + \left(1 - \frac{1}{\theta}\right)D$.)

(d) Show that if for the SOR method, $\lim_{k\to\infty} x_k = x^*$ for any $x_0 \in C^n$, then it is necessary that $\theta \in (0,2)$.

4. Computation of Eigenvalues and Eigenvectors

Let A be a nondefective $n \times n$ matrix with eigenvalues, $\lambda_1, \lambda_2, \dots, \lambda_n$, with $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$, and corresponding eigenvectors $u_1, u_2, \dots, u_n$. Let $x_0 \in C^n$ be such that $x_0 = \sum_{i=1}^{n} \alpha_i u_i$, with $\alpha_1 \neq 0$. Define the sequence of vectors $\{x_k\}_{k=1}^{\infty} \subseteq C^n$ recursively by $x_{k+1} = Ax_k$, $k = 0,1,2,\dots$

(a) Let $v \in C^n$ be any fixed vector that is not orthogonal to u_1. Show that $q_k = {v^T x_{k+1}}/{v^T x_k}$ converges to λ_1 as $k \to \infty$.

(b) Now suppose that $|\lambda_2| > |\lambda_3|$, $v \in C^n$ is orthogonal to u_1 but is not orthogonal to u_2 and $\alpha_2 \neq 0$. Show that $\lim_{k\to\infty} q_k = \lambda_2$.

(c) Now suppose $|\lambda_1| > |\lambda_2| > |\lambda_3| \geq |\lambda_4| \geq \cdots \geq |\lambda_n|$, $v \in C^n$ is such that $\alpha_1 v^T u_1 \neq 0$. Show that for k sufficiently large, $q_k \approx \lambda_1 + C(\lambda_2/\lambda_1)^k$ for some constant C. (Hint: Show that $\lim_{k\to\infty}(q_k - \lambda_1)(\lambda_1/\lambda_2)^k = C$, for some constant C.)

Numerical Analysis Preliminary Examination
Monday February 3, 2014

Work all problems and show all your work for full credit. This exam is closed book, closed notes, no calculator or electronic devices of any kind.

1. (a) Let $\{f_k\}_{k=1}^{n}$ be n linearly independent real valued functions in $L_2(a,b)$, and let Q be the $n \times n$ matrix with entries $Q_{i,j} = \int_a^b f_i(x) f_j(x) dx$. Show that Q is positive definite symmetric and therefore invertible.

   (b) Let g be a real valued functions in $L_2(a,b)$ and find the best (in $L_2(a,b)$) approximation to g in span$\{f_k\}_{k=1}^{n}$.

2. Let A be a 3×3 nonsingular matrix which can be reduced to the matrix

$$U = \begin{bmatrix} 1 & u_1 & u_2 \\ 0 & 1 & u_3 \\ 0 & 0 & 1 \end{bmatrix}$$

   using the following sequence of elementary row operations:

   (i) α_1 times Row 1 is added to Row 2.
   (ii) α_2 times Row 1 is added to Row 3.
   (iii) Row 2 is multiplied by $\frac{1}{\alpha_3}$.
   (iv) α_4 times Row 2 is added to Row 3.

   (a) Find an LU decomposition for the matrix A.

   (b) Let $b = [b_1 \quad b_2 \quad b_3]^T$ be an arbitrary vector in R^3 and let the vector $x = [x_1 \quad x_2 \quad x_3]^T$ in R^3 be the unique solution to the linear system $Ax = b$. Find an expression for x_3 in terms of the $\alpha_i{}'s$, the $b_i{}'s$, and the $u_i{}'s$, $i = 1,2,3$.

3. In this problem we consider the iterative solution of the linear system of equations $Ax = b$ with the following $(n-1) \times (n-1)$ matrices

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

(a) Show that the vectors $x^k = \left(\sin\frac{\pi k}{n}, \quad \sin\frac{2\pi k}{n}, \quad \cdots, \quad \sin\frac{\pi(n-1)k}{n}\right)$, for $k = 1, \cdots, n-1$ are eigenvectors of B_J, the Jacobi iteration matrix corresponding to the matrix A given above.

(b) Determine whether or not the Jacobi's method would converge for all initial conditions x^0.

(c) Let L and U be, respectively, the lower and upper triangular matrices with zero diagonal elements such that $B_J = L + U$, and show that the matrix $\alpha L + \alpha^{-1} U$ has the same eigenvalues as B_J for all $\alpha \neq 0$.

(d) Show that an arbitrary nonzero eigenvalue, λ, of the iteration matrix
$$H(\omega) = (I - \omega L)^{-1}\big((1-\omega)I + \omega U\big)$$
for the Successive Over Relaxation (SOR) method satisfies the following equation
$$\lambda^2 - 2(1-\omega)\lambda - \mu^2\omega^2\lambda + (1-\omega)^2 = 0,$$
where μ is an eigenvalue of B_J (Hint: use the result of (c)).

(e) For $n = 4$, find the spectral radius of $H(1)$.

4. (a) Find the singular value decomposition (SVD) of the matrix
$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}.$$

(b) Let $\{\lambda_k\}$ and $\{\sigma_k\}$ be the sets of eigenvalues and singular values of $n \times n$ matrix A. Show that: $\min_k \sigma_k \leq \min_k |\lambda_k|$ and $\max_k \sigma_k \geq \max_k |\lambda_k|$.

(c) Let A be a full column rank $m \times n$ matrix with singular value decomposition $A = U\Sigma V^*$, where V^* indicates the conjugate transpose of V .

(1) Compute the SVD of $A(A^*A)^{-1}A^*$ in terms of U, Σ, and V.

(2) Let $||\cdot|| = sup_{x\neq 0}\frac{|||Ax||_2}{||x||_2}$ be the matrix norm induced by the vector 2-norm, and let σ_{max} be the largest singular value of A. Show that $\big||A|\big| = \sigma_{max}$.

Preliminary Exam in Numerical Analysis Fall 2014

Instructions The exam consists of four problems, each having multiple parts. You should attempt to solve all four problems.

1. Linear systems

(a) Let A be an $n \times n$ matrix, B be an $n \times m$ matrix, C be an $m \times n$ matrix, and let D be an $m \times m$ matrix with the matrices A and $D - CA^{-1}B$ nonsingular. Show that the partitioned $(n+m) \times (n+m)$ matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is nonsingular and find an explicit formula for its inverse.

(b) How many multiplications would be needed to compute M^{-1} if you do **not** take advantage of its special structure.

(c) How many multiplications would be needed to compute M^{-1} if you do take advantage of its special structure (i.e. using the formulas you derived in part (a)).

2. Least squares

Let $A \in R^{m \times n}$ and $b \in R^m$, with $m \leq n$ and rank $A = m$. Let $x_0 \in R^n$ and consider the constrained optimization problem given by

$$\min ||x - x_0||,$$

subject to $Ax = b$, where the norm in the above expression is the Euclidean norm on R^n. Show that this problem has a unique solution given by

$$x^* = A^T(AA^T)^{-1}b + (I_n - A^T(AA^T)^{-1}A)x_0,$$

where I_n is the $n \times n$ identity matrix.

3. Iterative Methods

Consider the following matrix

$$A = \begin{bmatrix} -2 & \frac{1}{2} \\ -\frac{1}{2} & -2 \end{bmatrix}.$$

(a) Find a range for the real parameter ω such that Richardson's method for the solution of the linear system $Ax = b$,

$$x^{k+1} = x^k - \omega\left(Ax^k - b\right)$$

converges for any initial guess x^0.

(b) Find an optimal value for ω and justify your answer.

4. Computation of Eigenvalues and Eigenvectors

Consider the following matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1+\varepsilon \end{bmatrix}.$$

(a) Find a non-singular matrix $T(\varepsilon)$ such that the matrix $T^{-1}(\varepsilon)AT(\varepsilon)$ is in Jordan canonical form.

(b) Find the limit of $\|T(\varepsilon)\|$ as ε tends toward zero.

(c) Explain what this implies with regard to using the LR or QR methods to compute the eigenvalues of a matrix whose Jordan canonical form contains a Jordan block of size larger than 1.

Numerical Analysis Screening Exam, Fall 2015

FIRST NAME:

LAST NAME:

STUDENT ID NUMBER:

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PROBLEM 1 Consider the following 3×3 matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & \epsilon & \epsilon \\ 0 & \epsilon & \epsilon + \epsilon^2 \end{pmatrix}$$

(a) Use the Gram-Schmidt orthogonalization method on the columns of matrix A to derive a QR-decomposition of A.

(b) Use the Householder QR-factorization method to perform the same task as (a).

(c) Based on your calculations in (a) and (b), determine which method would lead to a more accurate factorization on an actual computer and justify your reasoning.

(d) A graduate student programmed both techniques in Matlab and tested them for the case $\epsilon = 10^{-10}$. He evaluated the norm of $A - Q * R$ in Matlab and found that the norm was equal to 0 for the Gran-Schmidt factorization and 4.6032×10^{-26} for the Householder factorization. Is this consistent with your conclusion in (c)? What other quantities would you suggest him to examine that may support your conclusion in (c)?

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PROBLEM 2. Let A be an $n \times n$ real-valued, symmetric positive definite matrix and $b \in \mathbb{R}^n$. Consider the following two-stage iterative procedure for solving the system of equations $Ax = b$:

$$\begin{aligned} x_{n+\frac{1}{2}} &= x_n + \omega_1(b - Ax_n) \\ x_{n+1} &= x_{n+\frac{1}{2}} + \omega_2(b - Ax_{n+\frac{1}{2}}) \end{aligned}$$

(a) Let $e_n = x - x_n$ be the error between the n-th iterate x_n and the exact solution x. Find the matrix K such that $e_{n+1} = Ke_n$.

(b) Find the eigenvalues of K in terms of the eigenvalues of A, ω_1, and ω_2.

(c) Show that ω_1 and ω_2 can be chosen so that the method converges with any initial condition. Express the rate of convergence in terms of λ_M and λ_m which correspond to the largest and smallest eigenvalues of A, respectively.

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PROBLEM 3. Consider a least square minimization problem:

$$\text{minimize}\|Ax - b\|_2^2, \qquad x \in \mathbb{R}^n, b \in \mathbb{R}^m, \tag{1}$$

and a regularized form of the (1):

$$\text{minimize}\|Ax - b\|_2^2 + \alpha\|x\|_2^2, \qquad \alpha > 0. \tag{2}$$

(a) State and justify a necessary and sufficient condition for a vector x_0 to be a solution of (1) and determine whether or not this problem always has a unique solution?

(b) State and justify a necessary and sufficient condition for a vector x_0 to be a solution of (2) and determine whether or not this problem always has a unique solution?

(c) Let $\mathcal{R}(A^T)$ be the range of A^T and let $\mathcal{N}(A)$ be the null space of A. Explain why a solution of (2) must be in $\mathcal{R}(A^T)$.

(d) Suggest a method for approximating a minimal norm solution of (1) using a sequence of solutions of (2) and justify your answer.

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PROBLEM 4. Let A be an $n \times n$ skew-Hermitian (i.e. $A^H = -A$) matrix.

(a) Show that $I + A$ is invertible.

(b) Show that $U = (I + A)^{-1}(I - A)$ is unitary.

(c) Show that if U is unitary with $-1 \notin \sigma(U)$, then there exists a skew-Hermetian matrix A such that $U = (I + A)^{-1}(I - A)$.

(d) Show that if B is an $n \times n$ normal matrix (i.e. $B^H B = BB^H$) then it is unitarily similar to a diagonal matrix.

(e) Let C be an $n \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$. Show that if λ is an eigenvalue of C, then $|\lambda| \leq \sigma_1$ and that $|\det(C)| = \prod_{i=1}^{n} \sigma_i$.

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Numerical Analysis Screening Exam
Fall 2016

Problem 1. (Iterative Methods) Consider the block matrix

$$A = \begin{bmatrix} I_n & 0 & -M_1 \\ -M_2 & I_n & 0 \\ 0 & -M_3 & I_n \end{bmatrix}$$

where I_n is the $n \times n$ identity matrix and M_1, M_2, and M_3, are $n \times n$ matrices.

a) Find the Jacobi and Gauss-Seidel iteration matrices for solving the system of equations $Ax = b$ (for some fixed vector b).
b) Show the methods either both converge or both diverge and, in the case of the former, the Gauss-Seidel method converges three times as fast as the Jacobi method.
c) Now consider the Jacobi and Gauss-Seidel methods for the system of equations $A^T x = b$. Show they either both converge or both diverge and, in the case of the former, the Gauss-Seidel method converges one-and-a-half times as fast as the Jacobi method.

Problem 2. (Least Squares) Let $A \in C^{m\times n}$ and consider the following set of equations in the unknown matrix $X \in C^{n\times m}$ known as the Moore-Penrose equations (MP):

$$AXA = A$$

$$XAX = X$$

$$(AX)^* = AX$$

$$(XA)^* = XA$$

(a) Show that the system (MP) has at most one solution (Hint: Show that if both $X$and Y are solutions to (MP) then $X = XAY$ and X=YAX.)

(b) When $A = \text{zeros}(m, n)$, show that there exists a solution to the system (MP). (Hint: Find one!)

(c) Assume that A has full column rank and find the solution to the least squares problem given by $\min\|AX - I\|_F^2$ where I denotes the $m \times m$ identity matrix and $\|\cdot\|_F$ denotes the Frobenius norm on $C^{m\times m}$ ($\|B\|_F = \left(\sum_{i=1}^m \sum_{j=1}^m b_{i,j}^2\right)^{\frac{1}{2}} = \left(\sum_{j=1}^m \sum_{i=1}^m b_{i,j}^2\right)^{\frac{1}{2}}$). (Hint: Note that the given least squares problem decouples into m separate standard least squares problems!)

(d) Assume that A has full column rank and use part (c) to show that the system (MP) has a solution. (Hint: Find one!).

(e) Assume $\text{rank}(A) = r$, $0 < r < n$, and show that there exists a permutation matrix $P \in C^{n\times n}$ such that $AP = [\hat{A}|\hat{A}R]$, or $A = [\hat{A}|\hat{A}R]P^T$ where $\hat{A} \in C^{m\times r}$ has full column rank and $R \in C^{r\times(n-r)}$. (A permutation matrix is a square matrix that has precisely a single 1 in every row and column and zeros everywhere else.)

(f) Assume $\text{rank}(A) = r$, $0 < r < n$, assume that a solution to the system (MP) has the form $X = P\begin{bmatrix} S \\ T \end{bmatrix}$, where $S \in C^{r\times m}$ and $T \in C^{(n-r)\times m}$ and use parts (c), (d), and (e) and the first equation in the system (MP) to determine the matrices $S \in C^{r\times m}$ and $T \in C^{(n-r)\times m}$ in terms of the matrix $\hat{A}$, and therefore show that the system (MP) has a solution.

Problem 3. (Direct Methods) Consider a vector norm $\|\cdot\|_V$ for $\mathbb{R}^n$. Define another norm $\|\cdot\|_{V^*}$ for $\mathbb{R}^n$ by $\|x\|_{V^*} = \max_{y\in\mathbb{R}^n, \|y\|_V \le 1} |x^T y|$.

It is known that for every vector $x \neq 0 \in \mathbb{R}^n$, there exists a vector $y \in \mathbb{R}^n$ such that $y^T x = \|y\|_{V^*}\|x\|_V = 1$. A vector y with this property is called a dual element of x.

a. Consider a nonsingular $n \times n$ matrix A. We define the distance between A and the set of singular matrices by

   $\mathrm{dist}(A) = \min\{\|\delta A\|_V : \delta A \in \mathbb{R}^{n\times n}, \text{ where } A + \delta A \text{ is singular}\}$,

   where $\|A\|_V$ is the operator norm induced by $\|\cdot\|_V$. Show that

$$\mathrm{dist}(A) \ge \frac{1}{\|A^{-1}\|_V}.$$

   Hint: Suppose the matrix $A + \delta A$ is singular. Then there exists $x \neq 0$ such that $(A + \delta A)x = 0$.

b. Let x be a unit vector such that $\|A^{-1}x\|_V = \|A^{-1}\|_V$ and $y = A^{-1}x/\|A^{-1}\|_V$. Consider a dual element z of y and the matrix

$$\delta A = -\frac{xz^T}{\|A^{-1}\|_V}.$$

   Show that $A + \delta A$ is singular.

c. Show that $\|\delta A\|_V = \|A^{-1}\|_V^{-1}$ and

$$\mathrm{dist}(A) = \frac{1}{\|A^{-1}\|_V}.$$

Problem 4. (Eigenvalue/Eigenvector Problems) Let A be a real symmetric matrix and q_1a unit vector. Let

$$\mathcal{K}(A, q_1, j) = \mathrm{Span}\{q_1, Aq_1, A^2 q_1, \dots, A^{j-1} q_1\}$$

be the corresponding Krylov subspaces. Suppose $\{q_1, q_2, \dots, q_n\}$ is an orthonormal basis for $\mathbb{R}^n$ and $\{q_1, q_2, \dots, q_j\}$ is an orthonormal basis for $\mathcal{K}(A, q_1, j)$, $1 \le j \le n$. Let Q_j be the $n \times j$ matrix whose columns are $q_1, q_2, \dots, q_j$, i.e. $Q_j = [q_1, q_2, \dots, q_j]$. It is easy to see that $Q_j^T A Q_j = T_j$ where T_j has the form

$$T_j = \begin{pmatrix} \alpha_1 & \beta_1 & 0 & & & \\ \beta_1 & \alpha_2 & \beta_2 & \cdots & & 0 \\ 0 & \beta_2 & \alpha_3 & & & \\ & \vdots & & \ddots & & \vdots \\ & 0 & & \cdots & \alpha_{j-1} & \beta_{j-1} \\ & & & & \beta_{j-1} & \alpha_j \end{pmatrix}$$

a) Derive an algorithm to compute the vectors q_j and the numbers α_j and β_j iteratively from the fact that $AQ_n = Q_n T_n$. This algorithm is known as the Lanczos algorithm.
b) Let M_j be the largest eigenvalue of T_j. Show that M_j increases as j increases and that M_n is equal to the largest eigenvalue of A, λ_1.
Hint: Recall that $M_j = \max_{x \in \mathbb{R}^j} x^T T_j x$ and $\lambda_1 = \max_{x \in \mathbb{R}^n} x^T A x$.
c) A standard approach for finding λ_1 is to use Householder transformations to tridiagonalize A and then use the QR algorithm. Suppose A is large and sparse. How could you use the Lanczos algorithm to improve on this method? What are the advantages of this alternative approach?

Exam in Numerical Analysis Spring 2015

Instructions The exam consists of four problems, each having multiple parts. You should attempt to solve all four problems.

1. **Linear systems**

Consider the symmetric positive definite (spd) $n \times n$ matrix A and its LU-decomposition $A = LU$ with $l_{ii} = 1$, $i = 1,2, \dots, n$.

a. Show that $u_{11} = a_{11}$, and $u_{kk} = \frac{\det(A_k)}{\det(A_{k-1})}$, $k = 2, \dots, n$, where for each $k = 1,2, \dots, n$, A_k is the $k \times k$matrix formed by taking the intersection of the first $k$rows and $k$columns of A.

b. Show that A can be written as $A = R^T R$ with R upper triangular with positive diagonal entries. (Hint: Let $D = \text{diag}(u_{11}, u_{22}, \dots, u_{nn})$ and consider the identity $A = LDD^{-1}U$.)

c. Show how the decomposition found in part (b) suggests a scheme for solving the system $Ax = b$ with A spd, that like Choleski's method requires *only* U, but that unlike Choleski's method, does *not* require the computation of square roots.

2. **Least squares**

Let $A \in R^{m \times n}$, $m \geq n$, be given.

(a) Show that the x-component of any solution of linear system

$$\begin{pmatrix} I_m & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} r \\ x \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix} , \tag{1}$$

is a solution of the minimization problem

$$\min_x ||b - Ax||_2. \tag{2}$$

(b) Show that the solution of the linear system (1) is unique if and only if the solution of the least squares problem (2) is unique.

3. **Iterative Methods**

Let A be an $n \times n$ nonsingular matrix and consider the matrix iteration

$$X_{k+1} = X_k + X_k(I - AX_k),$$

with X_0 given. Find and justify necessary and sufficient conditions on A and $X_0$for this iteration to converge to A^{-1}.

4. Computation of Eigenvalues and Eigenvectors

Consider the matrices A and T given by

$$A = \begin{bmatrix} 3 & \alpha & \beta \\ -1 & 7 & -1 \\ 0 & 0 & 5 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix},$$

where $|\alpha|, |\beta| \leq 1$.

(a) Use the similarity transform $T^{-1}AT$ to show that the matrix A has at least two distinct eigenvalues. (Hint: Gershgorin's Theorem)

(b) In the case $\alpha = \beta = 1$verify that $x = [1 \quad 1 \quad 1]^T$is an eigenvector of A and find a unitary matrix U such that the matrix $U^T AU$ has the form

$$\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}.$$

(c) Explain, without actually calculating, how one could find a Schur decomposition of A.

1. Suppose

$$A = \begin{pmatrix} D_1 & F_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ E_2 & D_2 & F_2 & 0 & \dots & 0 & 0 & 0 \\ 0 & E_3 & D_3 & F_3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & E_{n-1} & D_{n-1} & F_{n-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & E_n & D_n \end{pmatrix}$$

is a real block tridiagonal matrix where the blocks are all size $q \times q$ and the diagonal blocks D_i are all invertible, $1 \le i \le n$. Suppose, moreover, that A^t is block diagonally dominant, in other words

$$||D_i^{-1}||_1 \, (||F_{i-1}||_1 + ||E_{i+1}||_1) < 1$$

for $1 \le i \le n$ where $F_0 = E_{n+1} = 0$.

(a) Show A is invertible.

(b) Show A has a block LU decomposition of the form

$$A = \begin{pmatrix} I & 0 & 0 & \dots & 0 & 0 \\ L_2 & I & 0 & \dots & 0 & 0 \\ 0 & L_3 & I & \dots & 0 & 0 \\ \dots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I & 0 \\ 0 & 0 & 0 & \dots & L_n & I \end{pmatrix} \begin{pmatrix} U_1 & F_1 & 0 & \dots & 0 & 0 \\ 0 & U_2 & F_2 & \dots & 0 & 0 \\ 0 & 0 & U_3 & \dots & 0 & 0 \\ \dots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & U_{n-1} & F_{n-1} \\ 0 & 0 & 0 & \dots & 0 & U_n \end{pmatrix}$$

where

* $||L_i||_1 \le 1$, $2 \le i \le n-1$ and
* each matrix U_i is invertible with $||U_i||_1 \le ||A||_1$, $i \le i \le n$.

Hint: Recall, if a square matrix M has $||M||_1 < 1$ then $I - M$ is invertible and

$$(I - M)^{-1} = I + M + M^2 + M^3 + \dots$$

(c) Show how you can find this block LU decomposition numerically and how you can use it to solve the system of equations $Ax = b$ (for a given vector b). Explain the significance of the bounds in (b) and why this approach might be preferable to employing Gaussian elimination with pivoting on the whole of A.

2. Consider the following 2×3 matrix A

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 2 \end{pmatrix}.$$

(a) Find a two dimensional subspace S^* such that

$$\min_{x \in S^*, \|x\|_2 = 1} \|Ax\|_2 = \max_{\dim S = 2} \min_{x \in S, \|x\|_2 = 1} \|Ax\|_2.$$

Justify your answer.

(b) Find a rank one 2×3 matrix B such that $\|A - B\|_2$ is minimized and justify your answer.

3. Let X be a linear vector space over C and let P be the $n \times n$ matrix defined by the linear transformation on X^n given by

$$P\vec{x} = P[x_1, x_2, \dots, x_n]^T = [x_2, x_3, \dots, x_n, x_1]^T.$$

(a) What are the matrices P, P^0, P^2, P^{n-1} and P^n ? (Hint: Although you could do this with matrix multiplication, it's easier to base your answer on the underlying transformation.)

Let F be the $n \times n$ matrix given by $[F]_{j,k} = \frac{1}{\sqrt{n}}\overline{\omega}^{(j-1)(k-1)}, j, k = 1,2, \dots, n$, where $\omega = e^{\frac{2\pi i}{n}} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.

(b) Show that for $k = 1,2, \dots, n$, $P\vec{F}_k = \overline{\omega}^{(k-1)}\vec{F}_k$, where $\vec{F}_k$ is the k^{th} column of the matrix F.

(c) Show that the matrix F is unitary.

Let $a = \{a_i\}_{i=1}^n \subseteq C$, set $p_a(z) = \sum_{i=1}^n a_i z^{i-1}$ and let the $n \times n$ matrix A_a be given by

$$A_a = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & \dots & a_n \\ a_n & a_1 & a_2 & a_3 & \dots & a_{n-1} \\ a_{n-1} & a_n & a_1 & a_2 & \dots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_n & a_1 & \dots & a_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ a_2 & a_3 & a_4 & a_5 & \dots & a_1 \end{bmatrix}.$$

(d) Show that A_a is diagonalizable with eigenvalue/eigenvector pairs given by $\{p_a(\overline{\omega}^{(k-1)}), \quad \vec{F}_k\}$, $k = 1,2, \dots, n$. (Hint: Parts (a), (b) and (c).)

4. Let $\{z_k^0\}_{k=1}^n$ be n points in the complex plane and consider the following iteration:

$$z_k^{m+1} \text{ is equal to the average of} \begin{cases} z_k^m \text{ and } z_{k+1}^m & k = 1,2,\dots,n-1 \\ z_n^m \text{ and } z_1^m & k = n \end{cases}.$$

(a) Let $Z^m = [z_1^m, z_2^m, \dots, z_n^m]^T$ and rewrite the transformation from Z^m to Z^{m+1} given above in the form of a matrix iteration.

(b) Show that $\lim_{m\to\infty} z_k^m = \hat{z}$, $k = 1,2,\dots,n$, where $\hat{z} = \frac{1}{n}\sum_{j=1}^n z_j^0$. (Hint: The **RESULT** of Problem 3(d) might be of some help to you here. Note that you may use the result of problem 3(d) even if you were not able to prove it yourself.)

(c) What happens if, in parts (a) and (b) , the phrase "the average" in the definition of the iteration is replaced with "an arbitrary convex combination" ; that is:

$$z_k^{m+1} = \begin{cases} \alpha\, z_k^m + (1-\alpha)\, z_{k+1}^m & k = 1,2,\dots,n-1 \\ \alpha\, z_n^m + (1-\alpha)\, z_1^m & k = n \end{cases} \quad \text{for some } \alpha \in (0,1)?$$

Numerical Analysis Screening Exam
Spring 2017

1. Let $A \in \mathbb{C}^{n\times n}$ and let $A_j \in \mathbb{C}^n$ $j = 1,2,\ldots,n$ be the j^{th} column of A. Show that
$$|\det A| \le \prod_{j=1}^{n} \|A_j\|_1.$$
Hint: Let $D = diag(\|A_1\|_1, \|A_2\|_1, \ldots, \|A_n\|_1)$, and consider $\det B$, where $B = AD^{-1}$.

2. a) Let $A \in \mathbb{R}^{n\times n}$ be nonsingular and let $A_j \in \mathbb{R}^n$ $j = 1,2,\ldots,n$ be the j^{th} column of A. Use Gram-Schmidt to show that $A = QR$, where $Q$is orthogonal and $R$is upper triangular with $\|A_j\|_2^2 = \sum_{i=1}^{j} R_{i,j}^2 \qquad j = 1,2,\ldots,n.$

   b) Given $A, Q, R \in \mathbb{R}^{n\times n}$as in part (a) above with $A = QR$, and given $b \in \mathbb{R}^n$, perform an operation count (of multiplications only) for solving the linear system $Ax = b$.

3. Consider the constrained least squares problem:
$$* \begin{cases} \min\limits_x ||Ax - b||_2 \\ subject\ to\ \ Cx = d \end{cases}$$
where the $m \times n$ matrix A, the $p \times n$ matrix C, and the vectors $b \in \mathbb{R}^m$ and $d \in \mathbb{R}^p$ are given.

   a) Show that the unconstrained least squares problem
   $$\min_x ||Ax - b||_2$$
   is a special case of the constrained least squares problem *.

   b) Show that the minimum norm problem
   $$\begin{cases} \min\limits_x ||x||_2 \\ subject\ to\ \ Cx = d \end{cases}$$
   is a special case of the constrained least squares problem *.

   c) By writing $x = x_0 + Nz$, show that solving the constrained least squares problem * is equivalent to solving an unconstrained least squares problem
   $$** \min_z ||\tilde{A}z - \tilde{b}||_2 \ .$$
   What are the matrices N and $\tilde{A}$ and vectors x_0 and $\tilde{b}$?

   d) Use part c) to solve the constrained least squares problem * where

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 4 & 3 \\ 0 & 0 & 2 \\ 1 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 3 \\ 3 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 4 & 16 & 12 \\ -2 & -8 & -6 \\ 1 & 4 & 3 \\ -1 & -4 & -3 \end{bmatrix}, \qquad d = \begin{bmatrix} 12 \\ -6 \\ 3 \\ -3 \end{bmatrix}$$

4. Consider a stationary iteration method for solving a system of linear equations $Ax = b$ given by

$$y^k = x^k + \omega_0(b - Ax^k), \qquad x^{k+1} = y^k + \omega_1(b - Ay^k).$$

a) Show that the matrix B defined by $x^{k+1} = Bx^k + c$ has the form $B = \mu p(A)$ where $p(\lambda)$ is a second order polynomial in λ with leading coefficient equal to 1.

b) Show that the scaled Chebyshev polynomial $T_2(\lambda) = \lambda^2 - 1/2$ has the property that

$$\frac{1}{2} = \max_{-1 \le \lambda \le 1} |T_2(\lambda)| \le \max_{-1 \le \lambda \le 1} |q(\lambda)|$$

for all second order polynomial q with leading coefficient 1.

c) If we know that matrix A is Hermitian with eigenvalues in $(-1, 1)$, find coefficients ω_0 and ω_1 such that the proposed iterative scheme converges for any initial vector x^0.

d) What could you do if the eigenvalues of the matrix A is in $(\alpha, \beta)$to make the scheme convergent?

Numerical Analysis Screening Exam, Fall 2017

First Name:

Last Name:

There are a total of 4 problems on this test. Please start each problem on a new page and mark clearly the problem number and the page number on top of each page. Please also write your names on top of the pages.

I. Direct Methods (25 Points)

Suppose A and B are non-singular $n \times n$ matrices and the $2n \times 2n$ matrix

$$C = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

is also non-singular.

(a) Show that

$$\det(C) = \det(A)\det(A - BA^{-1}B)$$

(b) Show that both $A + B$ and $A - B$ are non-singular.

(c) Consider the system of equations $Cx = b$. Let

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

where x_1, x_2, b_1, b_2 are in $\mathbb{R}^n$. Let y_1 and y_2 be the solutions to

$$(A + B)y_1 = b_1 + b_2 \qquad \text{and} \qquad (A - B)y_2 = b_1 - b_2$$

Show that

$$x_1 = \frac{1}{2}(y_1 + y_2)$$
$$x_2 = \frac{1}{2}(y_1 - y_2)$$

(d) What is the numerical advantage of finding the solution to $Cx = b$ in this way?

II. Least Squares Problem (25 Points)

Consider the following least square optimization problem

$$\min_{x\in\mathbb{R}^3} \|Ax-b\|_2^2, \quad \text{where} \quad A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 4 \\ 1 & 1 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}.$$

(a) Show that $x = (3,1,0)^T$ is a solution to the least square problem.

(b) Find the minimum norm solution for this problem.

(c) Consider any vector $b = (b_1, b_2, b_3)^T$, show that a solution for the least square problem

$$\min_{x\in\mathbb{R}^3} \left\| \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 4 \\ 1 & 1 & 2 \end{pmatrix} x - b \right\|_2^2.$$

is given by

$$x = \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix}, \quad \text{where} \quad \alpha = \frac{b_1 + b_3 + 2b_2}{4}, \quad \beta = \frac{b_1 + b_3 - 2b_2}{4}.$$

(d) Using an approach similar to (b) to find the pseudo inverse of the matrix A.

III. Eigenvalue Problems (25 Points)

Consider the following matrix

$$A = \begin{pmatrix} -1 & 1 & 3 \\ 3 & -3 & 3 \\ 1 & 3 & -1 \end{pmatrix}.$$

(a) Show that $x = (1,1,1)^T$ is an eigenvector of A.

(b) Consider matrix Q given by

$$Q = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix}.$$

Show that $\hat{A} = Q^* A Q$ has same eigenvalues as matrix A and

$$\hat{A} = \begin{pmatrix} 3 & -2/\sqrt{6} & \sqrt{2} \\ 0 & -3 & \sqrt{3} \\ 0 & -1/\sqrt{3} & -5 \end{pmatrix}.$$

(c) Use the result of (b) to find a Schur decomposition of A .

(d) Try to perform one step of the LR algorithm on A. What transformation may be needed to compute the eigenvalues of matrix A using the LR algorithm?

IV. Iterative Methods (25 Points)

We want to solve the system

$$Ax = b$$

for an invertible matrix $A \in \mathbb{R}^{n \times n}$. Assume that we have a computational algorithm that computes an approximation $x_1 = \overline{A^{-1}}b$ of x. Ideally, we would choose $\overline{A^{-1}} = A^{-1}$ but let's say for example due to roundoff errors this is not available. In order to improve the result x_1 we can use the following strategy: For $\Delta := x - x_1$ we have

$$A\Delta = b - Ax_1.$$

Again we cannot compute this exactly but we can apply our algorithm $\overline{A^{-1}}$ to obtain $\bar{\Delta} = \overline{A^{-1}}(b - Ax_1)$. Then

$$x_2 = x_1 + \bar{\Delta} \approx x_1 + \Delta = x$$

is hopefully a better approximation of x. We can now iteratively apply this strategy to obtain a sequence x_k, $k = 1, 2, \ldots$.

(a) Rewrite this process as an iterative solver of the form $x_{n+1} = Bx_n + c$ for the system $Ax = b$.

(b) Show that for $\|\overline{A^{-1}} - A^{-1}\| < \|A\|$ this method converges to the correct solution x.

Numerical Analysis Screening Exam, Spring 2018

This exam has 4 problems. Please write down your names on all pages of your answer and mark clearly the order of pages. Start a new problem on a new page.

I. Direct Methods

Consider the matrix

$$A = \begin{bmatrix} R & \mid & v \\ u^T & \mid & 0 \end{bmatrix}$$

where R is an $n \times n$ invertible upper triangular matrix and u and v are vectors in $\mathbb{R}^n$.

(a) Consider the LU decomposition of A, $A = LU$, where L is unit lower triangular and U is upper triangular. Find the matrices L and U.

(b) Find necessary and sufficient conditions on R, u and v for A to be non-singular.

(c) Suppose A is non-singular. Use the LU-decomposition from (a) to formulate an economical algorithm for solving the system of equations $Ax = b$ and determine its computational cost.

II. Iterative Methods

Consider the system of equations $Ax = b$ where

$$A = D - E - F$$

is an $n \times n$ Hermitian positive definite matrix, D is diagonal, $-E$ is strictly lower triangular and $-F$ is strictly upper triangular. (In other words, both E and F have zeros on the diagonal.) To solve the system using the Gauss-Seidel method with successive over-relaxation (SOR) we write $A = M - N$ where

$$M = \frac{1}{\omega} D - E$$

$$\text{and} \qquad N = \left(\frac{1}{\omega} - 1\right) D + F$$

where ω is a real number. The iteration matrix is

$$T_\omega = M^{-1}N = (D - \omega E)^{-1} \left[(1 - \omega)D + \omega F\right].$$

(a) For any matrix B let B^* be its conjugate transpose. Show that if $|\omega - 1| < 1$ then $M^* + N$ is positive definite.

(b) Consider the vector norm $||\cdot||_A$ be defined by

$$||x||_A = \sqrt{x^* A x}.$$

and the corresponding matrix norm $||\cdot||_A$ it induces on $\mathbb{C}^{n\times n}$. Show that if $M^* + N$ is positive definite then $||M^{-1}N||_A < 1$.

(c) What can you deduce about the convergence of the SOR method in the case when $|\omega - 1| < 1$?

(d) Suppose the SOR method converges for all initial vectors x_0. Show that $|\omega - 1| < 1$. *(Hint: compute* $\det(T_\omega)$*.)*

III. Eigenvalues Problem

Suppose A is a real symmetric $n \times n$ matrix. Given two integers p and q between 1 and n, let

$$G = G(p,q,\theta) = \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & c & \dots & s & & \\ & & & \vdots & \ddots & \vdots & & \\ & & & -s & \dots & c & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \\ & & & & & & & & 1 \end{bmatrix}$$

be a Givens matrix. In other words, G is the $n \times n$ matrix defined by

$$\begin{array}{lll} G_{kk} = 1 \quad k \neq p, q & & G_{ij} = 0 \quad i \neq j \text{ and } (i,j) \neq (p,q) \text{ or } (q,p) \\ G_{pp} = c & & G_{pq} = s \\ G_{qq} = c & & G_{qp} = -s \end{array}$$

where $c = \cos\theta$ and $s = \sin\theta$.

(a) Let $A_1 = G^T A G$ and suppose θ is chosen so that $[A_1]_{pq} = 0$. Let $||\cdot||$ be the Frobenius norm given by

$$||M|| = \sum_{i,j} M_{ij}^2.$$

Let $A = D_0 + E_0$ and $A_1 = D_1 + E_1$ where D_0 and D_1 are diagonal matrices and E_0 and E_1 have zeros on their diagonals. Show that

$$||E_1||^2 = ||E_0||^2 - 2A_{pq}^2$$

(b) Use part (a) to describe an iterative method for finding the eigenvalues of A and prove your algorithm converges.

IV. Least Square Problem

Let $A \in \mathbb{R}^{m\times n}$ be a matrix and let $A = U\Sigma V^T$ be its singular value decomposition. We want to find k vectors $x_i \in \mathbb{R}^m$ and $y_i \in \mathbb{R}^n$ such that

$$\left\| A - \sum_{i=1}^{k} x_i y_i^T \right\|_F^2$$

is minimal, where $\|A\|_F^2 := \sum_{i=1}^m \sum_{j=1}^n a_{ij}^2$ is the Frobenius norm.

(a) Let u_i and v_j the i-th and j-th column of U and V respectively and σ_r the r-th singular value of A. Show that

$$x_i = \sqrt{\sigma_i} u_i, \qquad\qquad y_i = \sqrt{\sigma_i} v_i$$

is a solution of the optimization problem .

Hint: For any orthogonal matrix O and S we have $\|OAS\|_F = \|A\|_F$.

(b) Assume that $m = n$ is large and k small. How many values do you have to store for A and how many for its approximation $\sum_{i=1}^k x_i y_i^T$?

(c) Suppose that the matrix A represents a database of rankings of n movies by m movie viewers such that $a_{ij} \in \{0, \dots, 5\}$ is the star ranking of movie j by viewer i. Is it reasonable that A can be well approximated by some $\sum_{i=1}^k x_i y_i^T$ for some comparatively small k?

Hint: Discuss the case that there are k groups of users with similar interests and therefore similar movie rankings.

Numerical Analysis Screening Examination
Fall 2018-2019

1. (Numerical methods for finding eigenvalues) Let A be a square $n \times n$ matrix with eigenvalue-eigenvector pairs $\{(\lambda_i, u_i)\}_{i=1}^n$ satisfying $\{u_i\}_{i=1}^n$linearly independent and $|\lambda_1| > |\lambda_2| \geq \cdots . \geq |\lambda_n|$. Let $x_0 \in C^n$ be given with $x_0 = \sum_{i=0}^n \alpha_i u_i$ and $\alpha_1 \neq 0$. For $k = 1,2, \ldots$, set $\beta_k = x_{k-1}^T x_k / x_{k-1}^T x_{k-1}$, where $x_k = Ax_{k-1}$.
   a. Show that $\beta_k = \lambda_1 \left(1 + O\left(|\lambda_2/\lambda_1|^k\right)\right)$, as $k \to \infty$.
      (Recall that if $h > 0$, $r_k = O\left(h^k\right)$, as $k \to \infty$, if and only if there exists a positive integer k_0 and a positive constant M such that $|r_k| \leq Mh^k$, for all $k > k_0$, or, equivalently if and only if $\frac{|r_k|}{h^k}$ is bounded for all positive integers k.)
   b. Show that if the matrix A is symmetric, then $\beta_k = \lambda_1 \left(1 + O\left(|\lambda_2/\lambda_1|^{2k}\right)\right)$, as $k \to \infty$.
   c. Let α be a given complex number. Show how an iteration like the one given above can be used to find the eigenvalue of A that is closest to α.

2. (Iterative methods for linear systems) A square matrix A is said to be *power bounded* if all the entries in A^mremain bounded as $m \to \infty$.
   a. Show that if $\|A\| < 1$, where $\| \quad \|$ is some induced matrix norm, then A is power bounded.
   b. Establish necessary and sufficient conditions on the spectrum of a diagonalizable matrix A to be power bounded.
   c. For λ a complex number and ka nonnegative integer, let $J_k(\lambda)$ denote the $k \times k$matrix with λ's on the diagonal and 1's on the first super diagonal, and show that
   $$J_k(\lambda)^m = \sum_{j=0}^{k-1} \binom{m}{m-j} \lambda^{m-j} J_k(0)^j$$
   d. Find necessary and sufficient conditions for an arbitrary square matrix A to be power bounded.

3. (Least squares) Consider the following least square minimization problem
   $$\min_{x \in \mathbb{R}^4} \|Ax - b\|_2^2$$
   where
   $$A = \begin{pmatrix} 2 & 0 & 2 & 2 \\ 1 & 1 & 2 & -2 \\ 1 & 1 & 2 & -2 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}.$$
   a. Explain why the problem has a solution.
   b. Determine whether or not $x_0 = (1,1,0,0)$ is a solution.
   c. Determine whether or not x_0 is the minimum norm solution to this problem.
   d. Find the minimum norm solution.

Go on to page 2.

4. (Direct methods for linear systems)

a) Consider a block matrix

$$K = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

where E, F, G, and H are all square $n \times n$ matrices. Show that, in general,

$$\det(K) \neq \det(E)\det(H) - \det(F)\det(G)$$

but if either F or G (or both) is the zero matrix (so K is either block-lower or block-upper triangular) then

$$\det(K) = \det(E)\det(H).$$

b) Suppose that A is a non-singular $n \times n$ matrix and B is any $n \times n$ matrix such that the $2n \times 2n$ matrix

$$C = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

is also non-singular. By considering the matrix

$$\begin{bmatrix} I & 0 \\ -BA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

or otherwise, show that

$$\det(C) = [\det(A)]^2 \det(I - A^{-1}BA^{-1}B).$$

c) Now suppose that A and B are any $n \times n$ matrices such that the $2n \times 2n$ matrix C given in Part b is nonsingular. Use Part a to show that both of the matrices $A + B$ and $A - B$ must be non-singular.

d) Consider the system of equations $Cx = b$ where the matrix $C$is as given in Part c above. Let $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ where b_1, b_2 are in $\mathbb{R}^n$ and let y_1 and y_2 be the unique solutions to

$$(A + B)y_1 = b_1 + b_2 \quad \text{and} \quad (A - B)y_2 = b_1 - b_2$$

guaranteed to exist by Part c above. Show how to obtain the solution of $Cx = b$ from y_1 and y_2. What is the numerical advantage of finding the solution of $Cx = b$ in this way rather than finding it directly?

Numerical Analysis Prelim Spring 2019

January 11, 2019

Problem 1.
Let $\{\varphi_i\}_{i=1}^m$ be m linearly independent vectors in $\mathbb{R}^n$ and set $\Phi = [\varphi_1|\varphi_2|...|\varphi_m] \in \mathbb{R}^{n\times m}$.

(a) Prove that $\Phi^T\Phi$ is nonsingular

(b) Let $||\cdot||$ denote the Euclidean norm on $\mathbb{R}^n$, for $x \in \mathbb{R}^n$ define the map $P_\Phi : \mathbb{R}^n \to \mathbb{R}^n$ by $P_\Phi x = \varphi$, where $\varphi = \text{arg min}_{\psi\in\text{span}\{\varphi_i\}_{i=1}^m} ||x - \psi||$, and show that $P_\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is linear.

(c) A linear transformation P on a Hilbert space is said to be an orthogonal projection if 1) P is self adjoint (i.e. $P^* = P$) and 2) P is idempotent (i.e. $P^2 = P$). Show that $P_\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal projection.

(d) Find a set of vectors in $\mathbb{R}^n$ whose span is equal to the orthogonal complement of the subspace spanned by the $\{\varphi_i\}_{i=1}^m$.

Problem 2. For a given small value $\epsilon > 0$ consider a matrix A of the form

$$A = \begin{pmatrix} 1 & 1 \\ -\epsilon & \epsilon \end{pmatrix}.$$

a. Find the matrix operator norm $\|A\|_2$ induced by the Euclidean norm in $\mathbb{R}^2$. (Hint:$\|A\|_2 = \|Ax^*\|_2$ where $\|x^*\|_2 = 1$ and $\|Ax^*\|_2^2 \geq \|Ax\|_2^2$ for all $x \in \mathbb{R}^2$ with $\|x\|_2 = 1$.)

b. Find the matrix norm $\|A^{-1}\|_2$.

c. What is the smallest possible norm $\|\delta A\|_2$ for a matrix δA such that the matrix $A+\delta A$ is singular?

d. Find a matrix δA with the smallest possible norm $\|\delta A\|$ such that $A + \delta A$ is singular.

Problem 3. Consider a strictly diagonally dominant matrix $A = D - E - F$ where matrices D, E, F are diagonal, strictly lower-triangular, strictly upper-triangular matrices, respectively.

a. For any parameter $0 < \omega < 1$ and complex value λ with $|\lambda| \geq 1$, show that the matrix

$$A_{\lambda,\omega} = \omega^{-1}(1 - \omega - \lambda)D + F + \lambda E$$

has the same properties as matrix A; that is, that $A_{\lambda,\omega}$ is also a strictly diagonally dominant matrix. (Hint: Use the fact that $|1-\omega-\lambda| \geq |\lambda| - (1-\omega)$ to show that $|\omega^{-1}(1-\omega-\lambda)| \geq |\lambda|$)

b. Recall that the iteration matrix B_{SOR} for the Successive Over Relaxation (SOR) method is given by

$$B_{SOR} = (\omega^{-1}D - E)^{-1}((\omega^{-1}-1)D + F).$$

Show that the matrix $B_{SOR} - \lambda I$ is nonsingular for all $|\lambda| \geq 1$ and $0 < \omega < 1$.

c. Using the conclusion of part b above, what can be deduced about the convergence of the SOR method applied to a strictly diagonally dominant matrix A with $0 < \omega < 1$. Justify your answer.

Problem 4.

Consider the matrix

$$A = \begin{bmatrix} -2 & 10 & 100 & 200 \\ 0.01 & 5 & 100 & 1000 \\ .001 & .02 & 15 & 10 \\ 0 & 0 & .01 & 9 \end{bmatrix}$$

(a) Notice that A is close to upper triangular. From this observation alone, what do you expect to be true about its eigenvalues?

(b) Use Gerschgorin's theorem to locate the eigenvalues of A to within a region of the complex plane that is the union of four discs. Notice, the theorem does not do a very good job at locating the eigenvalues.

(c) Suppose T is invertible. Prove that the eigenvalues of $T^{-1}AT$ are the same as the eigenvalues of A.

(d) Find a diagonal matrix T such that when Gerschgorin's theorem is applied to $T^{-1}AT$, one obtains four disjoint disks in the complex plane, each of which contains an eigenvalue of A.

Numerical Analysis Preliminary Examination
Fall 2019

Problem 1.

(a) Prove that the product of two lower triangular matrices is lower triangular.
(b) Prove that the inverse of a nonsingular lower triangular matrix is lower triangular.
(c) Prove that if A is a nonsingular matrix that can be put in row echelon form using elementary row operations but without interchanging any rows, then it can be factored as $A = LU$ where L is lower triangular and U is upper triangular with all ones down the diagonal. Such a factorization is called an LU decomposition of the matrix.
(d) Prove if A is positive definite that it has an LU decomposition.

Problem 2. Suppose $A \in \mathbb{C}^{m\times m}$ is nonsingular. Consider the iterative scheme

$$X_{n+1} = X_n + c(AX_n - I)$$

to calculate A^{-1}, where $X_0 \in \mathbb{C}^{m\times m}$ and $c \in \mathbb{C}$, $c \neq 0$.

(a) Find a necessary and sufficient condition on A and c under which the scheme converges to A^{-1} for all initial matrices X_0.
(b) Suppose the eigenvalues of A are all real with

$$1 = \lambda_r \leq \lambda_{r-1} \leq \ldots \leq \lambda_1 = 5.$$

What is the optimal value of $c \in \mathbb{C}$ that achieves the maximum rate of convergence?
(c) Suppose now that the eigenvalues of A are all real with

$$-1 = \lambda_r \leq \lambda_{r-1} \leq \ldots \leq \lambda_1 = 5.$$

Show there is no value of c that will make the scheme converge for all initial matrices X_0. But consider the modified scheme

$$X_{n+1} = X_n + C(AX_n - I)$$

where C is now a matrix instead of a number. Find a matrix C such that this scheme converges to A^{-1} for all initial matrices X_0.

Problem 3. Let $\{\varphi_i\}_{i=1}^m$ be m linearly independent vectors in $\mathbb{R}^n$ and set

$$\Phi = [\varphi_1|\varphi_2|...|\varphi_m] \in \mathbb{R}^{n\times m}.$$

(a) Prove that $\Phi^T\Phi$ is nonsingular.
(b) Let $||\cdot||$ denote the Euclidean norm on $\mathbb{R}^n$ and define the map $P_\Phi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$P_\Phi x = \varphi, \quad \text{where} \quad \varphi = \arg \min_{\psi \in \text{span}\{\varphi_i\}_{i=1}^m} ||x - \psi||.$$

Show that P_Φ is a linear map.
(c) A linear transformation P on a Hilbert space is said to be an orthogonal projection if
   1) P is self adjoint (i.e. $P^* = P$) and
   2) P is idempotent (i.e. $P^2 = P$).

Show that P_Φ is an orthogonal projection.
(d) Find a set of vectors in $\mathbb{R}^n$ whose span is equal to the orthogonal complement of the subspace spanned by the $\{\varphi_i\}_{i=1}^m$.

Problem 4. The characteristic polynomial of a matrix $A \in \mathbb{C}^{5\times 5}$ has the form
$$p(\lambda) = (\lambda - 1)^3(\lambda - 2)^2.$$

(a) Find all possible Jordan canonical matrices J for A such that $J_{k,k} \geq J_{k+1,k+1}$, $k = 1\ldots, n-1$.
(b) For each of the matrices J find the algebraic and geometric multiplicity of its eigenvalues.
(c) For each of the matrices J find the minimal annihilating polynomial p_0 such that $p_0(A) = 0$.

Numerical Analysis Preliminary Examination
Spring 2020

Problem 1. Consider a matrix $A \in \mathbb{C}^{n\times m}$. The norm $||A||_F$ is given by

$$||A||_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m} |A_{i,j}|^2}.$$

Show that

$$||A||_F = \sqrt{\sum_{k=1}^{\min\{n,m\}} \sigma_k^2}$$

where σ_k, $k = 1, 2, \ldots, \min\{n, m\}$ are the singular values of the matrix A.

Problem 2. Suppose $A \in \mathbb{R}^{n\times n}$ is symmetric and positive definite and $b \in \mathbb{R}^n$. Consider the conjugate gradient method for finding the unique solution x^* to $Ax = b$. Let $x_0 \in \mathbb{R}^n$ be the initial vector used in the method and let $r_0 = b - Ax_0$. Suppose the method is carried out to infinite precision.

(a) Prove the method finds x^* in one step (in other words, $x_1 = x^*$) if and only if r_0 is either the zero vector or an eigenvector of A.

(b) Suppose A has $m \le n$ distinct eigenvalues. Show the method finds x^* in at most m iterations.

Problem 3. Let $A = A_{ij} \in \mathbb{C}^{m\times n}$ be a complex-valued matrix. The matrix $X \in \mathbb{C}^{n\times m}$ is said to satisfy the Moore-Penrose equations (MP) if:

$$AXA = A$$
$$XAX = X$$
$$(AX)^* = AX$$
$$(XA)^* = XA$$

(where, for $B \in \mathbb{C}^{m\times n}$, B^* denotes the complex conjugate transpose, $B^* = \bar{B}^T$).

(a) Given a matrix A, show there is at most one matrix X that satisfies (MP).

(b) Suppose $A = V\Sigma W^*$ is a singular value decomposition of A. Define $A^\dagger$ as follows.

- For a complex scalar λ, let

$$\lambda^\dagger = \begin{cases} \frac{1}{\lambda} & \lambda \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

- If A is *diagonal* (i.e. $A_{ij} = 0$ whenever $i \neq j$) define $A^\dagger \in \mathbb{C}^{n\times m}$ by

$$(A^\dagger)_{ij} = (A_{ji})^\dagger.$$

- Otherwise, define

$$A^\dagger = W\Sigma^\dagger V^*.$$

Show that $A^\dagger$ is a solution to (MP).

(c) For $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^m$, show that $x = A^\dagger b \in \mathbb{C}^n$ is the vector of minimum norm that minimizes $||Ax - b||_2$.

Problem 4. Consider the matrix

$$A = \begin{bmatrix} 2 & 10^2 & 10^2 \\ 10^{-2} & 1 & 10^2 \\ 10^{-2} & 10^{-2} & 2 \end{bmatrix}.$$

Show that the eigenvalues of A lie in $D_\varepsilon(1) \cup D_\varepsilon(2)$ where $\varepsilon = 10^{2/3} + 10^{-2/3}$ and the notation $D_\varepsilon(x)$ should be interpreted as

$$D_\varepsilon(x) = \{\lambda \in \mathbb{C} : |\lambda - x| < \varepsilon\}.$$

Hint: You might want to use a matrix of the form $T = \operatorname{diag}\{1, \alpha, \alpha^2\}$ *with a carefully selected value of* α*.*

Numerical Analysis Preliminary Examination Fall 2020

September 9, 2020

Problem 1.
Consider the normed linear space $\mathbb{C}^n$ with vector norm $\| \cdot \|$ and let $A, B \in \mathbb{C}^{n \times n}$. The definition of the matrix norm, $\| \cdot \|_M$, induced by the vector norm $\| \cdot \|$ is given by $\| A \|_M = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$.

(a) Show that for each $A \in \mathbb{C}^{n \times n}$ and each $x \in \mathbb{C}^n$, $\| Ax \| \leq \| A \|_M \| x \|$.

(b) Show that $\| A \|_M = \sup_{\|x\| \leq 1} \| Ax \|$.

(c) Show that in fact for each $A \in \mathbb{C}^{n \times n}$ there exists a $y \in \mathbb{C}^n$ with $\| y \| \leq 1$ such that $\| A \|_M = \| Ay \|$.

(d) Show that $\| AB \|_M \leq \| A \|_M \| B \|_M$.

(e) If $\| \cdot \| = \| \cdot \|_1$ find $\| \cdot \|_M$.

Problem 2.
Let $A \in \mathbb{C}^{m \times n}$ have full column rank.

(a) Find the mapping, P, of $\mathbb{C}^m$ onto the range of A defined by

$$Px = \text{argmin}_{y \in \text{Range}(A)} \| x - y \|_2^2 \quad x \in \mathbb{C}^m.$$

Hint: Use the normal equations.

(b) Show that P is self-adjoint and idempotent.

(c) Let $C, D \in \mathbb{C}^{m \times m}$, let $C = U \Sigma V^*$ be a singular value decomposition for C and suppose that $C = (I - 2P)D(I - 2P)^*$. Find a singular value decomposition for D. Justify your answer.

Problem 3.

(a) Consider the matrices

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ \delta & 1 \end{bmatrix}.$$

Show that for any $\varepsilon > 0$ and any real number K there exists δ such that $||A - B||_2 < \varepsilon$ and the eigenvalues μ of B have the property that for all eigenvalues λ of A,

$$|\mu - \lambda| \geq K||A - B||_2.$$

(b) Suppose now A is a real $n \times n$ symmetric matrix with eigenvalues λ_1, λ_2, …, λ_n and suppose $\mu \in \mathbb{R}$ and $u, w \in \mathbb{R}^n$ have the property that

$$(A - \mu I)u = w.$$

Show there is at least one eigenvalue λ_j such that

$$|\mu - \lambda_j| \leq \frac{||w||_2}{||u||_2}.$$

Hint: expand u and w in an appropriate basis.

(c) Use your result in b) to show that if A is a real symmetric matrix with eigenvalues λ_1, λ_2, …, λ_n and B is a perturbation of A that is also symmetric, then every eigenvalue μ of B has the property that there is at least one eigenvalue λ_j such that

$$|\mu - \lambda_j| \leq ||A - B||_2.$$

Hint: Let u be a corresponding eigenvector of B and write $(A - \mu I)u$ as $(A - B + B - \mu I)u$.

(d) Consider the matrix

$$A = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

Suppose this matrix is entered into the computer as

$$B = \begin{bmatrix} 1.0000 & 0.5000 & 0.3333 \\ 0.5000 & 0.3333 & 0.2500 \\ 0.3333 & 0.2500 & 0.2000 \end{bmatrix}$$

whose eigenvalues are

$$\mu_1 = 1.408294053$$
$$\mu_2 = 0.11223414532$$
$$\mu_3 = 0.002664493933$$

What can we say about where the eigenvalues of A lie?

Problem 4.

(a) Let Q be a symmetric matrix. Show that any two eigenvectors of Q, corresponding to distinct eigenvalues, are Q-conjugate.

(b) Let Q be a positive definite symmetric matrix and suppose $p_0, p_1, \ldots, p_{n-1}$ are linearly independent vectors in $\mathbb{R}^n$. Show that the Gram-Schmidt procedure can be used to generate a sequence of Q-conjugate directions from the p_i's. Specifically, show that $d_0, d_1, \ldots, d_{n-1}$ defined recursively by

$$d_0 = p_0$$
$$d_{k+1} = p_{k+1} - \sum_{i=0}^{k} \frac{p_{k+1}^T Q d_i}{d_i^T Q d_i} d_i$$

forms a Q-conjugate set.

(c) Let Q be a positive definite symmetric matrix and suppose p_0, p_1, …, p_{n-1} are linearly independent vectors in $\mathbb{R}^n$. Show how to find a matrix E such that $E^T Q E$ is diagonal.

Numerical Analysis Preliminary Examination
Spring 2021

January 22, 2021

Problem 1. Consider the normed linear space $\mathbb{C}^n$ with vector norm $\| \cdot \|_2$ and let $A \in \mathbb{C}^{n\times n}$. The matrix norm $\| \cdot \|$ induced by the vector norm $\| \cdot \|_2$ is defined as $\| A \| = \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}$.

(a) Show that $\| A \| = \sup_{\|x\|_2=1} \| Ax \|_2$.

(b) Show that in fact for each $A \in \mathbb{C}^{n\times n}$ there exists a $y \in \mathbb{C}^n$ with $\| y \|_2 = 1$ such that $\| A \| = \| Ay \|_2$.

(c) Use part (b) above and the eigenvalues $\{\lambda_j\}_{j=1}^n$ and corresponding eigenvectors $\{u_j\}_{j=1}^n$ of the matrix $A^H A \in \mathbb{C}^{n\times n}$ $(A^H = \bar{A}^T)$ to show that $\| A \| \leq \sqrt{\max_j \lambda_j}$.

(d) Show that in fact $\| A \| = \sqrt{\max_j \lambda_j}$.

Problem 2. Let A be an $m \times n$ matrix where $m > n$. We are particularly interested in the case when A does not have full rank. Suppose $b \in \mathbb{R}^m$ is a known vector. Consider the family of functions

$$\psi_\alpha(x) = ||b - Ax||_2^2 + \alpha||x||_2^2$$

where $\alpha > 0$ is a positive number.

(a) Find $\nabla\psi_\alpha$ and use it to derive the equivalent of the normal equations for finding the value(s) of x that minimize(s) ψ_α.

(b) Show that the minimum is attained at a unique vector $x = x_\alpha$.

(c) Let x^* be the solution of the least squares problem

$$\min_{x\in\mathbb{R}^n} ||b - Ax||_2^2$$

of minimum norm. Show that if $\alpha = \alpha_k \to 0$ as $k \to \infty$ then $x_{\alpha_k} \to x^*$.

Problem 3. Suppose A is an $n \times n$ symmetric positive definite matrix with corresponding A-inner product and A-norm defined by

$$\langle x, y \rangle_A = x^T A y \qquad \text{and} \qquad ||x||_A^2 = x^T A x.$$

Consider the system of equations $Ax = b$. Recall, the conjugate gradient method is an iterative method for finding the solution x^* that starts with a vector x_0 and finds iterates x_1, x_2, ... as follows:

$$\begin{aligned} r_i &= b - Ax_i \\ p_i &= r_i - \sum_{k<i} \frac{p_k^T A r_i}{||p_k||_A^2} p_k \\ x_{i+1} &= x_i + \frac{p_i^T r_i}{||p_i||_A^2} p_i \end{aligned}$$

Notice the definition of the p_i's uses Gram-Schmidt to ensure they are mutually A-orthogonal. For simplicity, you may assume in this problem that $r_i \neq 0$ for $i = 0, 1, \ldots, n-1$.

(a) Let x^* be the solution to $Ax = b$. Show

$$x^* = \sum_{i=0}^{n-1} \left(\frac{p_i^T b}{||p_i||_A^2} \right) p_i.$$

(b) Writing $x_0 = \sum_{i=0}^{n-1} \beta_i p_i$, show

$$x_k = \sum_{i=0}^{k-1} \left(\frac{p_i^T b}{||p_i||_A^2} \right) p_i + \sum_{i=k}^{n-1} \beta_i p_i$$

for $k = 1, 2, \ldots, n$.

(c) Use parts (a) and (b) to show x_k minimizes the function $\psi(x) = ||x - x^*||_A^2$ in the affine space

$$S_{k-1} = x_0 + \text{Span}\{p_0, p_1, \ldots, p_{k-1}\}$$

and deduce that r_k is orthogonal to $p_0, p_1, \ldots, p_{k-1}$ (with respect to the regular inner product).

(d) Show

$$\text{Span}\{r_0, r_1, \ldots, r_k\} = \text{Span}\{p_0, p_1, \ldots, p_k\} = \text{Span}\{r_0, Ar_0, \ldots, A^k r_0\}$$

for $k = 0, 1, \ldots, n-1$.

(e) Show that for each $k = 2, 3, \ldots, n-1$, r_k is A-orthogonal to p_i for $i \leq k-2$. What is the significance of this in terms of the execution of the conjugate gradient method?

Problem 4.

(a) Let $A = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$ be the $n \times n$ matrix A whose entries are all ones. Find all the eigenvalues and a full set of orthogonal eigenvectors of A. Make sure you explain how you got your answers. Hint: You might not want to use the determinant.

(b) One of the difficulties in calculating eigenvalues and eigenvectors is their sensitivity with respect to the entries in the matrix. For $\varepsilon \in \mathbb{R}$ Consider the following $n \times n$ matrix A_ε: ones on the first sub-diagonal, ε (a generally small nonzero number) in the upper right hand corner and zeros elsewhere. For example, when $n = 3$ the matrix A_ε is shown here: $A_\varepsilon = \begin{bmatrix} 0 & 0 & \varepsilon \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. What is the effect on the eigenvalues and eigenvectors of introducing this small nonzero entry ε into the matrix A_0 (i.e. the matrix A_ε with $\varepsilon = 0$).

Numerical Analysis Preliminary Examination Fall 2021

September 2, 2021

Problem 1.
Given a function of time, $g \in C(0,1)$, consider the inverse filtering, or deconvolution, problem of determining a function of time $f \in L_2(0,1)$, that satisfies $Hf = g$, where the linear operator H is the convolution operator on $L_2(0,1)$ given by

$$(Hf)(t) = \int_0^t h(t-s)f(s)ds, \quad 0 \le t \le 1,$$

with the kernel, or filter, $h \in C(0,1)$ known and given. Let V be an n dimensional subspace of $L_2(0,1)$ with basis functions $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$, let $\{t_1, t_2, \ldots, t_m\}$ be specified with $0 \le t_1 < t_2 < \ldots < t_m \le 1$ and $m > n$, and let g and h denote respectively, the given function and kernel in $C(0,1)$.

(a) Formulate the problem of finding a least squares solution $f^{m,n}$ in the subspace $V \subset L_2(0,1)$ to the approximating discretized inverse filtering problem given by

$$(Hf^{m,n})(t_i) = g(t_i), \quad i = 1, 2, \ldots, m,$$

in the form $\min_{\beta \in \mathbb{R}^n} \| A^{m,n}\beta - b^m \|^2$. That is, what are the matrix $A^{m,n}$ and the vector b^m?

(b) What is a necessary and sufficient condition on the matrix $A^{m,n}$ for the least squares problem formulated in part (a) to have a unique solution?

(c) If the condition in part (b) is not satisfied, what is the solution of minimum norm to the least squares problem in part (a) in terms of the singular values of the matrix $A^{m,n}$?

(d) If we define the row vector of functions $\Phi^n \in L_2^n(0,1)$ by $\Phi^n = [\varphi_1, \varphi_2, \ldots, \varphi_n]$, then any $\varphi^n \in V \subset L_2(0,1)$ can be written as $\varphi^n = \Phi^n\beta$, for some vector $\beta \in \mathbb{R}^n$. Show that $\| \varphi^n \|^2_{L_2(0,1)} = \beta^T M^n \beta$ for some matrix $M^n \in \mathbb{R}^{n\times n}$, and show that the matrix M^n is positive definite and symmetric.

(e) Now let $\lambda > 0$ be given, and formulate the approximating discretized inverse filtering problem with Tychonov regularization of minimizing over V (or equivalently, over $\mathbb{R}^n$) the functional

$$J(\beta;\lambda) = \| A^{m,n}\beta - b^m \|^2 + \lambda \| \varphi^n \|^2_{L_2(0,1)}$$

as a least squares problem of the form $\min_{\beta\in\mathbb{R}^n} \| \hat{A}^{m,n}(\lambda)\beta - \hat{b}^{m,n} \|^2$ (i.e. what are the matrix $\hat{A}^{m,n}(\lambda)$ and vector $\hat{b}^{m,n}$?) where $A^{m,n}$ and b^m are as they were defined in part (a) and φ^n and β are as they were defined in part (d).

(*f*) Verify that the least squares problem defined in part (e) has a unique solution and then find it.

Problem 2. Consider the vector space $\mathbb{R}^n$ endowed with the l_2-norm $||\cdot||_2$ and let $||\cdot||$ be the induced norm on $n\times n$ matrices. Let A be an $n\times n$ invertible matrix.

(a) Explain how we know that $\inf_{||v||_2=1}||Av||_2$ is attained and denote the unit vector that attains it by x.

(b) Let $y = Ax$. Show $||A^{-1}|| = \frac{1}{||y||_2}$.

(c) Give a geometric description of the action of the matrix yx^T and find $||yx^T||$. *Hint: what is the image of a vector $z = \alpha x + w$ where w is orthogonal to x?*

(d) Consider the matrix $B = A - yx^T$. Show B is singular.

(e) Let

$$\Delta = \inf\{||\delta A|| : A + \delta A \text{ is singular}\}$$

denote the distance of A from the set of singular matrices. Show

$$\Delta = \frac{1}{||A^{-1}||}.$$

Problem 3. Suppose A is a real $n\times n$ symmetric and positive definite matrix with eigenvalues

$$0 < \lambda_n \le \lambda_{n-1} \le \ldots \le \lambda_1$$

and $b\in\mathbb{R}^n$. The Richardson Iteration Method for finding the solution to $Ax = b$ is

$$x_{k+1} = x_k - \omega(Ax_k - b)$$

where ω is an iteration parameter.

(a) For what values of ω is the Richardson Iteration Method guaranteed to converge for any starting iterate x_0?

(b) Find the value of ω that optimizes the rate of convergence.

(c) Consider the matrix

$$A = \begin{bmatrix} 2+a & -1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ -1 & 2+a & -1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -1 & 2+a & -1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & -1 & 2+a & -1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & -1 & 2+a \end{bmatrix}$$

where $a > 0$. Use Gerschgorin's Theorem to find upper and lower bounds for the eigenvalues of A.

(d) If you want to solve $Ax = b$ using the Richardson Iteration Method for the matrix A in (c), use your answers to (b) and (c) to determine a good choice for ω. Explain and give a bound for the number of iterations sufficient to reduce the norm of the matrix by a factor of 10^{-6}.

Problem 4.

(a) Consider a matrix that has real eigenvalues and n linearly independent eigenvectors x_i, and the largest eigenvalue in magnitude λ_1 is dominant. That is $(|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|)$. Show that if the vector v_0 has a nonzero component α_1 in the direction of x_1 then $\lim_{m\to\infty} \left(\frac{1}{\lambda_1^m}\right) A^m v_0 = \alpha_1 x_1$

Also, $\lambda_1 = \lim_{m\to\infty} \frac{y^T A^{m+1} v_0}{y^T A^m v_0}$ where y is a vector not orthogonal to x_1.

(b) Get an approximation for x_1 and λ_1 by doing three iterations of the method in part (a). Take $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and start from $v_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Comment on your method and results.

(c) A Householder transformation is a matrix of the form:

$$H = I - 2\frac{vv^T}{||v||^2}.$$

Show that H is symmetric and orthogonal. Furthermore, show that if

$$z = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \sigma = ||x||, \ v = x + \sigma\ z \text{ Then } Hx = -\sigma\ z$$

(d) Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Using part (c) find an orthogonal matrix U such that $U^{-1}AU$ is tridiagonal. How can this help us to compute the eigenvalues of A?. Check that $U^{-1}AU$ is tridiagonal. Hint: The matrix H will be imbedded into the lower right corner of the matrix U

Numerical Analysis Preliminary Examination Fall 2022

August 25, 2022

Problem 1.
(a) Show that if $A \in \mathbb{C}^{n\times n}$ is strictly diagonally dominant -i.e., $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$, $i = 1, 2, ..., n$ then A is nonsingular.

(b) Provide a nontrivial example, (that is a matrix with no zero rows) of a diagonally dominant matrix that is singular.

(c) Suppose that $A \in \mathbb{C}^{n\times n}$ is strictly diagonally dominant with $a_{ii} \in \mathbb{R}$ and $a_{ii} > 0$, $i = 1, 2, ..., n$. Show that every eigenvalue of A has positive real part.

(d) Let $A \in \mathbb{C}^{n\times n}$, and suppose that there is a $k \in \{1, 2, \ldots, n\}$ for which $|a_{kk}| \geq \sum_{j\neq k} |a_{kj}|$ and that $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$ for $i \neq k$. Show that A is nonsingular. (Hint: Apply Gershgorin's Theorem to the matrix $B = D^{-1}AD$, where D is an appropriately chosen diagonal matrix).

Problem 2.
Show that if $A \in \mathbb{C}^{n\times n}$ is diagonalizable with $A = E\Lambda E^{-1}$, $B \in \mathbb{C}^{n\times n}$ and $\hat{\lambda}$ is an eigenvalue of $A + B$, then there is an eigenvalue λ of A such that $|\lambda - \hat{\lambda}| \leq \kappa(E)\|B\|_\infty$, where $\kappa(E)$ denotes the condition number of the matrix E with respect to the infinity matrix norm, $\|\cdot\|_\infty$, on $\mathbb{C}^{n\times n}$.

Problem 3.
Let x be the least squares solution to the problem $Ax = b$, where A is a $m \times n$ real matrix of rank n.

(a) Derive the normal equations and write the solution x as $x = A^\dagger b$ where $A^\dagger$is the pseudo-inverse of A. That is find a compact expression for $A^\dagger$ in terms of A.

(b) Let $\tilde{b} = b + \delta b$ be a perturbation of the vector b. The matrix A remains unchanged. Let $\tilde{x}$ be the least squares solution to $A\tilde{x} = \tilde{b}$. Show that if $b_R \neq 0$, then $\frac{\|\tilde{x}-x\|}{\|x\|} \leq \operatorname{cond}(A)\frac{\|\delta b_R\|}{\|b_R\|}$, where $\operatorname{cond}(A) = \|A\|\|A^\dagger\|$, b_R and δb_R are, respectively, the projections of the vectors b and δb onto $R(A)$.

In the following questions, give your answers in terms of fractions and square roots and whenever A, b or δb are mentioned, use:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \delta b = 10^{-3} \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

(c) Use Gauss elimination to find the least squares solution of $Ax = b$.

(d) Find and use the Cholesky decomposition to find the least squares solution of $Ax = b$.

(e) Use part (b) to get an upper bound on $\frac{\|\tilde{x}-x\|}{\|x\|}$. Use any induced norm you like.

(f) Find and use a full or reduced QR decomposition of A to find the least squares solution of $Ax = b$. Use your favorite method to find this QR decomposition.

(g) Find and use the SVD of A to find the least squares solution of $Ax = b$. Use your favorite method to find this SVD.

Numerical Analysis Preliminary Examination
Spring 2023

January 8, 2023

Problem 1. (20 points)

Suppose $A \in \mathbb{R}^{n\times n}$ is symmetric and let $\|\cdot\|_F$ denote the Frobenius norm on $\mathbb{R}^n$ defined by

$$\|B\|_F^2 = \sum_{i,j=1}^{n} B_{ij}^2.$$

Given two integers $1 \le p < q \le n$ and an angle θ, let $c = \cos\theta$ and $s = \sin\theta$ and consider the matrix $G = G(p,q,\theta)$ for which

$$\begin{array}{llll} G_{kk} = 1 & k \neq p,q & G_{ij} = 0 & i \neq j \text{ and } (i,j) \neq (p,q) \text{ or } (q,p) \\ G_{pp} = c & & G_{pq} = -s & \\ G_{qq} = c & & G_{qp} = s & \end{array}$$

This matrix is shown below.

$$G = G(p,q,\theta) = \begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & \ddots & & & & & & \\ & & & c & \dots & -s & & & \\ & & & \vdots & \ddots & \vdots & & & \\ & & & s & \dots & c & & & \\ & & & & & & \ddots & & \\ & & & & & & & 1 & \\ & & & & & & & & 1 \end{bmatrix}$$

Consider the matrix $B = G^T A G$. Notice, if $i,j \neq p,q$ then $B_{ij} = A_{ij}$, if $i \neq p,q$ then

$$\begin{aligned} B_{ip} &= cA_{ip} + sA_{iq} \\ B_{iq} &= -sA_{ip} + cA_{iq}, \end{aligned}$$

if $j \neq p, q$ then

$$\begin{aligned} B_{pj} &= cA_{pj} + sA_{qj} \\ B_{qj} &= -sA_{pj} + cA_{qj}, \end{aligned}$$

and

$$\begin{aligned} B_{pp} &= c^2 A_{pp} + 2csA_{pq} + s^2 A_{qq} \\ B_{pq} &= B_{qp} = (c^2 - s^2)A_{pq} + cs(-A_{pp} + A_{qq}) \\ B_{qq} &= c^2 A_{qq} - 2csA_{pq} + s^2 A_{pp}. \end{aligned}$$

Jacobi's iterative method for finding the eigenvalues of A starts with the matrix $A^{(0)} = A$. For $k = 1, 2, \ldots$, values of $p_k < q_k$ and θ_k are chosen and the matrix $A^{(k)} = G^T A^{(k-1)} G$ is constructed.

(a) (5 points) Show that, for all k, $A^{(k)}$ is symmetric, $\|A^{(k)}\|_F^2 = \|A\|_F^2$, and the eigenvalues of $A^{(k)}$ are the same as the eigenvalues of A.

(b) (5 points) Given values of p_k and q_k, find a value of θ_k so that $A^{(k)}_{p_k q_k} = 0$.

(c) (5 points) Let $A^{(k)} = D^{(k)} + E^{(k)}$ where $D^{(k)}$ is diagonal and $E^{(k)}$ has zeros on the diagonal. If θ_k is chosen as in part (b) then it is a straightforward calculation to see that

$$||E^{(k)}||_F^2 = ||E^{(k-1)}||_F^2 - 2\left(A^{(k-1)}_{p_k q_k}\right)^2.$$

(You do not need to show this.) Show that $p_k < q_k$ can be chosen at each stage so that

$$\|E^{(k)}\|_F^2 \leq \left(1 - \frac{2}{n^2}\right)^k \|E^{(0)}\|_F^2.$$

(d) (5 points) Let $\varepsilon > 0$ be given. Show there exists K such that for all $k \geq K$, every eigenvalue of A lies within ε of a diagonal element of $A^{(k)}$ and every diagonal element of $A^{(k)}$ lies within ε of an eigenvalue of A.

Problem 2. (20 points)

(a) (5 points) Suppose $B \in \mathbb{R}^{n\times n}$ and consider the sequence defined by

$$x^{(k+1)} = Bx^{(k)} + d$$

where $d \in \mathbb{R}^n$. Show the sequence converges for all d and all $x^{(1)}$ to a limit that is independent of $x^{(1)}$ if and only if $B^k \to 0$. *Hint. Show that the matrix* $I - B$ *is invertible if* $B^k \to 0$.

(b) (5 points) Suppose $B \in \mathbb{R}^{n\times n}$. Show that B^k converges to zero as k goes to infinity is equivalent to the spectral radius of B being less than 1. *Hint. Write* $SBS^{-1} = J$, *where* J *is the Jordan form of* B:

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_r \end{bmatrix} \qquad \text{where} \qquad J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda i & 1 & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}.$$

Also, write $J_i = \lambda_i I + N$ *where* N *is a nilpotent matrix and use the binomial theorem on* J_i.

(c) (5 points) Let $Ax = b$, where $A \in \mathbb{R}^{n\times n}$ is a non singular matrix. Use (b) to show that if A is strictly row diagonally dominant then the Jacobi method converges for any arbitrary choice of the initial value $x^{(1)}$. *Hint: Recall that, for any induced norm,* $\rho(B) \leq \|B\|$ *where* $\rho(B)$ *denotes the spectral radius of* B.

(d) (5 points) Let

$$A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix}.$$

Apply two iterations of the Jacobi method to the system $Ax = b$ starting at $x^{(1)} = (0, 0, 0)^T$.

Problem 3. (20 points)

For $n = 1, 2, ...,$ let $\{\varphi_j^n\}_{j=0}^n \subset C[0,1]$ be given by $\varphi_0^n(x) = 1 - nx$ if $x \in [0, \frac{1}{n}]$, $\varphi_0^n(x) = 0$ otherwise, $\varphi_n^n(x) = nx - n + 1$ if $x \in [\frac{n-1}{n}, 1]$, $\varphi_n^n(x) = 0$ otherwise, and for $j = 1, 2, ..., n-1$, $\varphi_j^n(x) = nx - j + 1$ if $x \in [\frac{j-1}{n}, \frac{j}{n}]$, $\varphi_j^n(x) = j + 1 - nx$ if $x \in [\frac{j}{n}, \frac{j+1}{n}]$, $\varphi_j^n(x) = 0$ otherwise.

(a) (2 points) Sketch the graphs of φ_j^n for $j = 0, 1, 2, ..., n$

(b) (4 points) Compute the Gramian matrix, M^n, considered as vectors in $L_2(0,1)$, $M_{i,j}^n = \langle \varphi_i^n, \varphi_j^n \rangle_{L_2(0,1)}$ of $\{\varphi_j^n\}_{j=0}^n$.

(c) (3 points) Let $S = \text{span}\{\varphi_j^n\}_{j=0}^n$ and use part (b) to argue that $\{\varphi_j^n\}_{j=0}^n$ is in fact a basis for the subspace $S \subset L_2(0,1)$.

(d) (3 points) Let $\varphi \in C[0,1]$ be given and find an expression for $I^n\varphi \in S$, where I^n denotes the interpolation operator on $C[0,1]$ with respect to the mesh $\{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1\}$ on $[0,1]$; that is, $(I^n\varphi)(\frac{j}{n}) = \varphi(\frac{j}{n})$, $j = 0, 1, 2, ..., n$.

(e) (3 points) Let $\varphi \in L_2(0,1)$ be given and let P^n be the orthogonal projection of $L_2(0,1)$ onto S. Then $P^n\varphi = \sum_{j=0}^n b_j\varphi_j^n \in S$, for some $\mathbf{b} = [b_0, b_1, ..., b_n]^T \in \mathbb{R}^{n+1}$. What is (i.e. compute or provide an expression for) $\mathbf{b}$?

(f) (2 points) Given that for $\varphi \in C^2(0,1)$, $\|I^n\varphi - \varphi\|_{L_2(0,1)} \leq \frac{K_0}{n^2}\|D^2\varphi\|_{L_2(0,1)}$ for some constant K_0, argue that $\|P^n\varphi - \varphi\|_{L_2(0,1)}$ is also $O(\frac{1}{n^2})$ for $\varphi \in C^2(0,1)$.

(g) (3 points) Use (1) the fact that for $\varphi \in C^2(0,1)$, $\|DI^n\varphi - D\varphi\|_{L_2(0,1)} \leq \frac{K_0}{n}\|D^2\varphi\|_{L_2(0,1)}$, (2) Part (f) above, and (3) the Schmidt inequality which states that

$$\int_a^b |Dp_k(x)|^2 dx \leq \frac{C_k}{(b-a)^2} \int_a^b |p_k(x)|^2 dx, \quad k = 1, 2, 3, or$$

where p_k is a polynomial of degree k and C_k is a constant that depends only on k, and not on a, b or p_k to argue that $\|DP^n\varphi - D\varphi\|_{L_2(0,1)}$ is also $O(\frac{1}{n})$ for $\varphi \in C^2(0,1)$. Note: D denotes the differentiation operator, $D = \frac{d}{dx}$. (Hint: Use the fact that $\int_0^1 = \sum_{j=1}^n \int_{\frac{j-1}{n}}^{\frac{j}{n}}$ and the triangle inequality and consider $\|DP^n\varphi - DI^n\varphi\|_{L_2(0,1)}$.)

Numerical Analysis Preliminary Examination Fall 2023

August 9, 2023

Problem 1.

Let $T \in \mathbb{C}^{n\times n}$ be given and let $\rho(T)$ denote the spectral radius of T.

(a) Show that for any induced norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$, $\|T\| \geq \rho(T)$.

(b) Given $\varepsilon > 0$, show there exists an induced norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$ for which $\|T\| \leq \rho(T)+\varepsilon$.

Hint:

- *Recall that if $\|\cdot\|_*$ is an induced norm on matrices and S is invertible then the matrix function defined by $\|A\| = \|SAS^{-1}\|_*$ is also an induced norm. (You do not need to show this.)*
- *First show this the result is true when T is a Jordan block*

$$T = J_n(\lambda) = \begin{bmatrix} \lambda & 1 & \dots & 0 & 0 \\ 0 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & \dots & 0 & \lambda \end{bmatrix}$$

by considering the norm defined by $\|A\| = \|D(n,\varepsilon)AD(n,\varepsilon)^{-1}\|_1$ where

$$D(n,\varepsilon) = \begin{bmatrix} 1/\varepsilon & 0 & \dots & 0 \\ 0 & 1/\varepsilon^2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1/\varepsilon^n \end{bmatrix}.$$

- *Then show it is true for general $T \in \mathbb{C}^{n\times n}$.*

(c) Show that for any induced norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$,

$$\lim_{k\to\infty} \|T^k\|^{1/k} = \rho(T).$$

Problem 2.

Consider the following $Ax = b$ Least Squares Problem (LSP).

$$\begin{bmatrix} 1 & -4 \\ 2 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 15 \\ 9 \end{bmatrix}$$

For each of the following questions, set up the equations needed to solve the problem, evaluate the matrices and vectors involved and work a few calculations to illustrate that you know the steps required to solve the problem. It is not necessary to perform all the calculations.

(a) Solve the LSP above using the Normal equations.

(b) Use the Gram-Schmidt method to find a full QR factorization of A and use this factorization to solve the LSP.

(c) Use Householder reflectors to find a full QR factorization of A. *Hint: a Householder reflector H is a matrix of the form $H = I - 2vv^T/v^Tv$ where $v \neq 0$.* Are the factorizations found in parts (b) and (c) necessarily equal? Explain.

Problem 3.

Let $A \in \mathbb{C}^{m\times n}$ and consider the following system of matrix equations in the unknown matrix $X \in \mathbb{C}^{n\times m}$:

$$\begin{aligned} AXA &= A \\ XAX &= X \\ (AX)^* &= AX \\ (XA)^* &= XA, \end{aligned} \tag{1}$$

where $*$ denotes conjugate transpose.

(a) Show that if the system (1) has a solution, $X = A^+ \in \mathbb{C}^{n\times m}$, then it is unique.

(b) Use the singular value decomposition of A to show that in fact there exists a (unique) solution $A^+ \in \mathbb{C}^{n\times m}$ to the system (1).

(c) Show that if $b \in \mathbb{C}^m$ and $\mathrm{rank}(A) = n$, then $x = A^+b$ is the least squares solution to the system $Ax = b$.

(d) Show that if the system $Ax = b$ has at least one solution, then $x = A^+b$ is the solution of minimum Euclidean norm.

Numerical Analysis Preliminary Examination Spring 2024

December 14, 2023

Problem 1. Suppose $A \in \mathbb{R}^{n\times n}$ is symmetric and positive definite.

(a) Show that all the diagonal entries in A are positive and that the entry in A with largest absolute value lies on the diagonal of A.

Suppose we perform Gaussian elimination on the matrix A. Let $A^{(0)} = A$ and let $A^{(k)}$ be the $(n-k)\times(n-k)$ matrix in the lower right corner after k rounds.

(b) Show that each $A^{(k)}$ is symmetric positive definite.

Hint: Write $A = A^{(0)}$ *as*

$$A = \begin{bmatrix} \alpha & v^T \\ v & B \end{bmatrix}$$

and show that

$$A^{(1)} = B - \frac{vv^T}{\alpha}$$

and that for every $x = [x_2, x_3, \ldots, x_n]^T \in \mathbb{R}^{n-1}$*, if* $y = [y_1, x_2, \ldots, x_n]^T$ *and* y_1 *is suitably chosen , then* $x^T A^{(1)} x = y^T A y$*.*

(c) What is the significance of (b) to the execution of Gaussian elimination on the matrix A?

(d) Consider the norm on $n \times n$ matrices defined by $\|C\| = \max_{i,j} |C_{ij}|$ and let U be the upper triangular matrix we obtain from doing Gaussian elimination on A. Show

$$\|A^{(k)}\| \leq \|A\|$$

for $k = 1, 2, \ldots, n-1$ and that $\|U\| \leq \|A\|$.

(e) What is the significance of (d) to the execution of Gaussian elimination on the matrix A?

Problem 2. The following questions are related to eigenvalues and eigenvectors of a non-defective $n \times n$ matrix A.

(a) Let $A = (a_{ij})$ be such a matrix and let $r_i = \sum_{j \neq i} |a_{ij}|, i = 1, \ldots, n$. Show that for each eigenvalue λ of A at least one of the following inequalities hold:

$$|\lambda - a_{ij}| \leq r_i \qquad i = 1, \ldots, n$$

(b) Apply the results in part a) to the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

(c) Let λ be an eigenvalue of A. Show that for any induced norm of A we have: $|\lambda| \leq \|A\|$.

(d) Let $|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|$. That is λ_1 is the dominant eigenvalue of A. Show that for appropriate starting vectors x_0, the iteration $x_k = A^k x_0$ can be used to approximate λ_1 and its corresponding eigenvector. Furthermore, show that the rate of convergence is determined by the ratio $|\frac{\lambda_2}{\lambda_1}|$.

(e) Apply the method in part (d) to the matrix A in part (b). Do only two iterations.

Problem 3. Consider the stationary vector-matrix iteration given by

$$x_{k+1} = Mx_k + c \tag{1}$$

where $M \in \mathbb{C}^{n \times n}$, $c \in \mathbb{C}^n$, and $x_0 \in \mathbb{C}^n$ are given.

(a) If $x^* \in \mathbb{C}^n$ is a fixed point of (1) and $\|M\| < 1$ where $\|\cdot\|$ is any compatible matrix norm induced by a vector norm, show that x^* is unique and that $\lim_{k \to \infty} x_k = x^*$ for any $x_0 \in \mathbb{C}^n$.

(b) Let $\rho(M)$ denote the spectral radius of the matrix M and use the fact that $\rho(M) = \inf \|M\|$, where the infimum is taken over all compatible matrix norms induced by vector norms, to show that $\lim_{k \to \infty} x_k = x^*$ for any $x_0 \in \mathbb{C}^n$ if and only if $\rho(M) < 1$.

(c) Now consider the linear system

$$Ax = b \tag{2}$$

where $A \in \mathbb{C}^{n \times n}$ is nonsingular and $b \in \mathbb{C}^n$ are given. What are the matrix $M \in \mathbb{C}^{n \times n}$ and the vector $c \in \mathbb{C}^n$ in (1) in the case of the Jacobi iteration for solving the linear system given in (2)?

(d) Use Part (a) to show that if the matrix $A \in \mathbb{C}^{n \times n}$ is row diagonally dominant then the Jacobi iteration will converge to the solution of the linear system given in (2).

(e) Use Part (b) together with the Gershgorin Circle Theorem to show that if the matrix $A \in \mathbb{C}^{n \times n}$ is row diagonally dominant then the Jacobi iteration will converge to the solution of the linear system given in (2).