1. Let $X_1, \dots, X_n$ be i.i.d. from a normal distribution with unknown mean μ and variance 1. Suppose that negative values of X_i are truncated at 0, so that instead of X_i, we actually observe

$$Y_i = \max(0, X_i), \qquad i = 1, 2, \dots, n,$$

from which we would like to estimate μ. By reordering, assume that $Y_1, \dots, Y_m > 0$ and $Y_{m+1} = \dots = Y_n = 0$.

(a) Explain how to use the EM algorithm to estimate μ from $Y_1, \dots, Y_n$. Specifically, give the details about E-step and M-step. Show that a recursive formula for the successive EM estimates $\mu^{(k+1)}$ is

$$\mu^{(k+1)} = \frac{1}{n}\sum_{i=1}^{m} Y_i + \frac{n-m}{m}\mu^{(k)} - \frac{n-m}{m}\frac{\phi(\mu^{(k)})}{\Phi(-\mu^{(k)})},$$

where $\phi(x)$ is probability density function and $\Phi(x)$ is cumulative density function of the standard normal distribution.

(b) Find the log-likelihood function $\log L(\mu)$ based only on observed data, and use it to write down a (nonlinear) equation which the MLE $\widehat{\mu}$ satisfies.

(c) Use the equation in part (b) to verify that $\widehat{\mu}$ is indeed a fixed point of the recursion found in (a).

(d) Prove that $\mu^{(k)} \longrightarrow \widehat{\mu}$ for any starting point $\mu^{(0)}$, providing at least one of the observations is not truncated. To do this, prove that the difference between $\mu^{(k)}$ and $\widehat{\mu}$ gets smaller as k gets larger. *Hint:* The Mean Value Theorem and the following inequalities, which you can use without proof, might be useful.

$$0 < \frac{\phi(x)[\phi(x) - x\Phi(-x)]}{\Phi^2(-x)} < 1, \qquad \text{for all } x.$$

Note: The Mean Value Theorem says that if f is continuous and differentiable on the interval (a, b), then there is a number c in (a, b) such that $f(b) - f(a) = f'(c)(b - a)$.

2. Let $X_1, \dots, X_n$ be iid Unif$(0, \theta)$, where $\theta > 0$ is unknown.

(a) Find the MLE $\widehat{\theta}$, its c.d.f. $F_\theta(u) = P_\theta(\widehat{\theta} \le u)$, and its expected value $E_\theta(\widehat{\theta})$.

(b) Consider a confidence interval for θ of the form

$$[a\widehat{\theta}, b\widehat{\theta}], \quad \text{where } 1 \le a \le b \text{ are constants.} \tag{1}$$

For given $0 < \alpha < 1$, characterize all $1 \le a \le b$ making $[a\widehat{\theta}, b\widehat{\theta}]$ a $(1-\alpha)$ confidence interval.

(c) Find values $1 \le a \le b$ minimizing the expected length $E_\theta(b\widehat{\theta} - a\widehat{\theta})$ among all $(1-\alpha)$ confidence intervals of the form (1), uniformly in θ.