ODE EXAM - Spring 2023

The exam has **four** problems on **two** pages. Each problem is worth 10 points. Do all four problems.

Your work should be neat and well organized. Neatness will not be officially taken into account in the scoring, but a greater degree of clarity will allow the committee to more confidently evaluate your work.

1. Let $a : [0, \infty) \to [0, \infty)$ and $u : [0, \infty) \to [0, \infty)$ be two nonnegative continuous functions. Assume that

$$u(x) \leq \int_0^x a(y)u(y)dy$$

for all $x \geq 0$. Show, *without citing* Gronwal's inequality, that $u(x) = 0$ for $x \geq 0$. To clarify, you cannot simply claim that the result follows from Gronwal's inequality; instead, you either *establish* Gronwal's inequality in this setting or use some other argument.

2. Consider the 2^{nd} order ODE for the unknown function $x = x(t)$,

$$x'' + p(t)x' + ax = 0,$$

where $p(t) = 2 - 3\cos(t)$ and a is a real number. Suppose $\phi(t)$ and $\psi(t)$ form a fundamental set of solutions, i.e. the matrix

$$X(t) = \begin{pmatrix} \phi(t) & \psi(t) \\ \phi'(t) & \psi'(t) \end{pmatrix}$$

is non-singular. Prove that

$$\lim_{t \to +\infty} \det X(t) = 0,$$

that is, the determinant of the matrix $X(t)$ converges to zero as $t \to +\infty$.

3. Consider the system for the unknown functions $x(t)$ and $y(t)$:

$$\begin{cases} x' = -y - y^2 x^3 \\ y' = x - x^2 y. \end{cases} \tag{1}$$

(i) Identify the stationary points of the system.

(ii) Prove that the system (1) has a unique solution $(x(t), y(t))$ satisfying $x(0) = 1,\ y(0) = 0$, and the solution is defined for all $t \geq 0$.

(iii) Prove that the solution from part (ii) satisfies

$$\lim_{t \to +\infty} x(t) = \lim_{t \to +\infty} y(t) = 0.$$

4. Consider the two-dimensional ODE

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}), \tag{2}$$

where the vector field $\boldsymbol{f}$ is continuously differentiable everywhere in $\mathbb{R}^2$. Suppose Γ is a periodic orbit for (2).

(i) What can we conclude about the index of Γ with respect to $\boldsymbol{f}$? Give a short explanation.

(ii) What can we conclude about the number and type of stationary points of $\boldsymbol{f}$ inside the region enclosed by Γ? Provide as many details as you can.

(iii) Which of the following statements about the stationary points of $\boldsymbol{f}$ inside the region enclosed by Γ are definitely NOT true? Explain your conclusions.

1. $\boldsymbol{f}$ has exactly one stationary point inside the region enclosed by Γ, and the point is a saddle.
2. $\boldsymbol{f}$ has exactly two stationary points inside the region enclosed by Γ: a saddle and a center.
3. $\boldsymbol{f}$ has exactly three stationary points inside the region enclosed by Γ: a saddle, a center, and a stable node.