PARTIAL DIFFERENTIAL EQUATIONS QUALIFYING EXAM
Spring 2023

Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Start each problem on a fresh sheet of paper and write on only one side of the paper.

1. Consider the equation
$$u_x^2(x,y) + 2u_y^2(x,y) = x^2 + 2y^2,$$
   (a) Find at least one classical solution to this equation in $\mathbb{R}^2$ such that $u(x,x) = x^2$.
   (b) Is the problem from (a) uniquely solvable?

2. Let u be harmonic in the domain $U = B(0,4)$ in $\mathbb{R}^2$.
   (a) Show that if u is bounded then
$$\sup_{x\in U}(4 - |x|)|\nabla u(x)| < \infty;$$
   (b) Give an example of u that is harmonic in U and unbounded, but still satisfies the claim from (a).

3. Suppose u is a C^2 function satisfying
$$\begin{cases} u_{tt} = \Delta u \quad \text{in } \mathbb{R}^n \times \mathbb{R}^+, \\ u(x,0) = g(x), \\ \partial_t u(x,0) = h(x). \end{cases}$$
Fix $T > 0$ and consider the set
$$K_T := \{(x,t) : 0 \le t \le T, |x| \le T - t\}.$$
Prove that, if $g(x) = h(x) = 0$ for all $x \in B(0,T)$, then $u(x,t) = 0$ for all $(x,t) \in K_T$.