1. Let $X_1, ..., X_n$ be an i.i.d. sample from the uniform distribution $U[0, \theta]$, $\theta > 0$.

   (a) Find a sufficient statistic for θ and construct a pivotal quantity for θ based on this sufficient statistic. Finally, construct a lower $(1-\alpha)$-confidence bound $\widehat{\theta}_L$ for θ based on this pivotal quantity, that is, $P(\theta \geq \widehat{\theta}_L) \geq 1 - \alpha$.

   (b) Show that the family of uniform distributions defined above has the monotone likelihood ratio property (with respect to which statistic?)

   (c) Find the most powerful test of size $\alpha = 0.1$ for testing $H_0 : \theta = \theta_0$ against $H_a : \theta > \theta_0$ for some $\theta_0 > 0$. Invert the rejection region of the test to construct a confidence interval for θ. Compare your result to part (a).

2. Let $X_1, \ldots, X_n$ be i.i.d. normal $N(0, \sigma^2)$ for some $\sigma^2 > 0$.

   (a) Find a maximum likelihood estimator (MLE) of σ^2.

   (b) Find a MLE of σ (without the square!), denoted $\widehat{\sigma}_n$, and find the asymptotic distribution (as $n \to \infty$), including the asymptotic variance, of $\sqrt{n}(\widehat{\sigma}_n - \sigma)$ using the general asymptotic properties of the MLE.
   (hint: you can use the fact that $\mathbb{E}X_1^4 = 3\sigma^4$)

   (c) Assume that the sample is of size $n = 2$ and $X_1 = -2$, $X_2 = 4$. Find the upper 75% confidence bound for σ using the non-parametric bootstrap. When would the bootstrap approach be advantageous, compared to using the asymptotic pivotal quantity constructed in part (b)?