1. Let $X_1, ..., X_n$ be an i.i.d. sample from the exponential distribution $\mathsf{Exp}(\lambda)$ with probability density function

$$f(x; \lambda) = \lambda e^{-\lambda x}, \quad x > 0,$$

parametrized with $\lambda > 0$ (or the "rate parameter") which we are tasked to estimate. Let $n > 1$.

   (a) Show that the maximum likelihood estimator of λ is $\widehat{\lambda}_n = \frac{n}{\sum_{j=1}^n X_j}$.

   (b) Show that $\widehat{\lambda}_n$ is biased. On average, does it overestimate or underestimate λ? You can assume that $\mathbb{E}\widehat{\lambda}_n$ is finite.
   (hint: the function $f(x) = \frac{1}{x}$ has some properties that could be useful in this question)

   (c) Show that (for example, using δ-method)

$$\sqrt{n}(\widehat{\lambda}_n - \lambda) \overset{n\to\infty}{\longrightarrow} N(0, \sigma_\infty^2)$$

   for some $\sigma_\infty^2 > 0$, where $\overset{n\to\infty}{\longrightarrow}$ denotes convergence in distribution. Find σ_∞^2.

2. Answer the following questions (a correct answer with incorrect justification is worth 0 points):

   (a) Let $X_1, \ldots, X_n$ be i.i.d. random variables generated according to some distribution from the family $P_\theta,\ \theta \in \Theta \subseteq \mathbb{R}$, where the value of the parameter θ is unknown. Give a definition of a sufficient statistic. Does a sufficient statistic always exist?

   (b) Prove or disprove: a Uniform Minimal Variance Unbiased estimator always exists.
   (hint: consider a sample of size 1 from the Bernoulli distribution with parameter $p \in [0, 1]$)

   (c) Prove or disprove: a maximum likelihood estimator is always unbiased.