Geometry and Topology Graduate Exam
Spring 2023

Solve as many problems as you can. Partial credit will be given to partial solutions.

Problem 1. Let X be a Hausdorff topological space, and let $\pi : \widetilde{X} \to X$ be its universal cover, i.e. $\widetilde{X}$ is path connected and simply connected and π is a covering map. Prove that if $\widetilde{X}$ is compact then the fundamental group of X is finite.

Problem 2. Let A be an $n \times n$ matrix which is symmetric and nonsingular, and let c be a nonzero real number. Prove that

$$\{x \in \mathbb{R}^n \mid \langle Ax, x\rangle = c\}$$

is a smooth submanifold of $\mathbb{R}^n$, and state its dimension. Here $\langle -, -\rangle$ denotes the standard inner product on $\mathbb{R}^n$.

Problem 3. Let $\omega \in \Omega^2(M)$ be an exact 2-form on a manifold M. Prove that for any map $f\colon S \to M$ from a closed orientable surface (i.e., closed orientable 2-dimensional manifold) S, there must be some $p \in S$ such that $(f^*\omega)_p = 0$.

Problem 4. Let $\mathbb{T}^2 = S^1 \times S^1$ denote the standard 2-torus and S^2 the standard 2-sphere. Let X be the space obtained by identifying two distinct points a_1, a_2 from $\mathbb{T}^2$ to some point $p \in S^2$. Compute (1) the (integral) homology groups of X in every degree, and (2) the fundamental group of X.

Problem 5. Let $n > 1$, let $\mathbb{T}^n = (\mathrm{S}^1)^n$ denote the n-torus, and let S^n denote the standard unit sphere in $\mathbb{R}^{n+1}$.

(a) Let $f : \mathbb{T}^n \to S^n$ be a smooth map satisfying the following properties:
- there exists 5 mutually disjoint open subsets $U_1, \dots, U_5$ of $\mathbb{T}^n$ such that for each i $f|_{U_i}$ is a diffeomorphism from U_i onto the open southern hemisphere $S^n \cap \{x_{n+1} < 0\}$;
- The image of the complement of these subsets lies in the northern hemisphere; that is $f(\mathbb{T}^n - \cup_i U_i) \subset S^n \cap \{x_{n+1} \geq 0\}$.

(You may take for granted such an f exists). Show that the induced map on nth de Rham cohomology $f^* : H^n_{dR}(\mathrm{S}^n) \to H^n_{dR}(\mathbb{T}^n)$ must be non-zero.

(b) Show that there does *not* exist a continuous map $f\colon \mathrm{S}^n \to \mathbb{T}^n$ from the n-sphere to the n-torus $\mathbb{T}^n = (\mathrm{S}^1)^n$ inducing a non-zero map $f_*\colon \mathrm{H}_n(\mathrm{S}^n) \to \mathrm{H}_n(\mathbb{T}^n)$ of n-th homology groups.

Problem 6. Let X be a topological space. Suppose for some k that we can cover X by k open sets $U_1, \dots, U_k$ so that each U_i is contractible as is each higher intersection of s open sets $U_{i_1} \cap \dots \cap U_{i_s}$ for every s. Prove that the reduced homology $\tilde{H}_i(X) = 0$ for all $i \geq k - 1$.

Problem 7. Let X denote the vector field on $\mathbb{R}^3$ given in standard coordinates by $X = x_1 \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_2} + 3x_3 \frac{\partial}{\partial x_3}$, and let $\phi_t : \mathbb{R}^3 \to \mathbb{R}^3$ denote the induced flow (you may take for granted that this exists for all time and is an oriented diffeomorphism).

If $R = [0,1]^3$ denotes the unit cube, compute the rate of change of the (standard) volume of $\phi_t(R)$ at $t = 0$. That is, compute:

$$\frac{d}{dt}(\int_{\phi_t(R)} dx_1 dx_2 dx_3)_{t=0}.$$

Hint: Re-express the above integral as an integral of a form which is varying in t, over a region that is not varying in t. You may also use the fact that $\frac{d}{dt}\int_A(\omega_t) = \int_A \frac{d}{dt}(\omega_t)$.