REAL ANALYSIS GRADUATE EXAM

Spring 2023

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let f be Lebesgue measurable in $[0,1]$ with $f(x) > 0$ for a.e. x. Suppose E_k is a sequence of measurable sets in $[0,1]$ with the property $\int_{E_k} f(x)\,dx \to 0$ as $k \to \infty$. Prove that $m(E_k) \to 0$ as $k \to \infty$.

2. Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions on $[0,1]$ such that
(i) $|g_n(x)| \leq C$ a.e. $x \in [0,1]$ for some $C < \infty$, and
(ii) $\lim_{n\to\infty} \int_0^a g_n(x)\,dx = 0$ for all $a \in [0,1]$.
Prove that for all $f \in \mathcal{L}^1([0,1])$ one has

$$\lim_{n\to\infty} \int_0^1 f(x)g_n(x)\,dx = 0.$$

3. Suppose E is a measurable subset of $[0,1]$ with the Lebesgue measure $m(E) = \frac{99}{100}$. Show that there exists a number $x \in [0,1]$ such that for all $r \in (0,1)$ one has

$$m(E \cap (x-r, x+r)) \geq \frac{r}{4}.$$

(Hint: Use the Hardy-Littlewood inequality $m(\{x \in \mathbb{R} : Mf(x) \geq \alpha\}) \leq \frac{3}{\alpha}\|f\|_{\mathcal{L}^1}$, where Mf denotes the Hardy-Littlewood maximal function of f.)

4. Let $f\colon [0,1] \to [0,1]$ be Lebesgue measurable. Prove that for every $M > 0$, there exists $a \in [0,1]$ such that

$$\int_0^1 \frac{dx}{|f(x)-a|} \geq M.$$