

**COMPLEX ANALYSIS GRADUATE EXAM**  
**Spring 2023**

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let  $u(z)$  be a bounded harmonic function in  $\mathbb{D} = \{|z| < 1\}$  such that we have the limits

$$\lim_{r \rightarrow 1^-} u(re^{i\varphi}) = \begin{cases} 1 & \text{if } 0 < \varphi < \pi \\ 0 & \text{if } \pi < \varphi < 2\pi. \end{cases}$$

Find  $u(\frac{1}{2})$ .

2. Let  $\Gamma$  be a closed curve in the right half plane that has index  $N$  with respect to the point 1. Find

$$\int_{\Gamma} e^{\frac{1}{z^2-1}} \sin \pi z \, dz.$$

3. Let  $\mathcal{F}$  be that family of power series  $\sum_{n=1}^{\infty} a_n z^n$  for which  $|a_n| \leq n$  for all  $n \in \mathbb{N}$ . Is  $\mathcal{F}$  a normal family in the open unit disk  $\mathbb{D} = \{|z| < 1\}$ ?

4. Let  $r \in (0, 1)$ . Prove that  $p_n(z) = 1 + 2z + 3z^2 + \dots + nz^{n-1}$  has no zeros in  $\{z \in \mathbb{C} : |z| < r\}$  provided  $n$  is sufficiently large.