Algebra Exam January 14, 2023

Show your work. Be as clear as possible. Do all problems.

1. Let $R = \mathbb{C}[x, y, x]/(z^2 - xy)$.
   (a) Show that R is an integral domain.
   (b) Show that R is integrally closed (hint: identify R as an integral extension of a polynomial ring).

2. Let G be a finite group and let p be the smallest prime divisor of $|G|$. Assume that a Sylow p-subgroup P of G is cyclic.
   (a) Show that $N_G(P) = C_G(P)$ (hint: what is $\text{Aut}(P)$?).
   (b) Show that if G is solvable, then G contains a subgroup N of index p.
   (c) Show that if N is a subgroup of index p (whether or not G is solvable), then N is normal in G.

3. Let F be a field extension of $\mathbb{Q}$ with $[F : \mathbb{Q}] = 60$ and $F/\mathbb{Q}$ Galois. Prove that if F contains a 9th root of 1, then $F/\mathbb{Q}$ is a solvable.

4. Let R be a finite ring with 1. Show that some element of R is not the sum of nilpotent elements. Give an example to show that 1 can be a sum of nilpotent elements.

5. Let F be an algebraically closed field with $A \in M_n(F)$.
   (a) Show that there exist polynomials $f(x), g(x) \in F[x]$ so that $A = f(A) + g(A)$ with $f(A)$ diagonalizable and $g(A)$ nilpotent.
   (b) Assuming (a), show that if $A = S+N$ with S diagonalizable, N nilpotent and $SN = NS$, then $S = f(A)$ and $N = g(A)$ (in otherwards, S and N are unique).

6. Let R be a ring with 1. Let M be a noetherian (left) R-module.
   (a) Show that if $f : M \to M$ is a surjective R-module homomorphism, then f is an isomorphism.
   (b) Show that if $f : M \to M$ is an injective R-module homomorphism, it need not be an isomorphism.