**MATH 507a** **QUALIFYING EXAM** **Spring 2023**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems.

(1) Let $X_1, X_2, \ldots$ be independent (not necessarily iid) with positive finite variance, and $S_n = \sum_{k=1}^n X_k$. Let $v_n = \mathrm{Var}\,(\mathrm{S_n})$. Suppose that for some $q > 2$,

$$\lim_n v_n^{-q/2} \sum_{k=1}^n E(|X_k - EX_k|^q) = 0. \tag{1}$$

Show that $v_n^{-1/2}(S_n - ES_n)$ converges in distribution to a standard normal.

(2) Let $X_1, X_2, \ldots$ be exponential r.v.'s with parameter $\lambda$ (that is, density $\lambda e^{-\lambda x}, x \geq 0$.) Let $S_n = \sum_{k=1}^n kX_k$.

(a) Find the mean and variance of $S_n$.

(b) Show that there is a constant $c$ such that $S_n/n^2 \to c$ in probability, and find $c$. HINT: $\sum_{k=1}^n k = n(n+1)/2$, $\sum_{k=1}^n k^2 = n(n+1)(2n+1)/6$.

(3) Let $\mu$ be a probability measure on $\mathbb{R}$ and $\varphi$ its characteristic function.

(a) For $a \in \mathbb{R}$, express

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-ita} \varphi(t)\, dt$$

in terms of the measure $\mu$.

(b) If $\lim_{|t|\to\infty} \varphi(t) = 0$, show that $\mu$ has no atoms (that is, no values $x$ with $\mu(\{x\}) > 0$.)