MATH 505a **QUALIFYING EXAM** **Spring 2023**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems. When a problem asks you to find something, you are expected to simplify the answer as much as possible.

(1) Suppose that each of k jobs is assigned at random to one of four servers A, B, C, and D. "At random" here means that there are 4^k equally likely outcomes.

Find the probability of the event E = (every server gets at least one job).

(2) Let X_n have a Poisson distribution with parameter n. Find constants a_n such that $\sqrt{X_n} - a_n$ converges in distribution, and find the limiting distribution.

HINT: What is the limit in distribution of $(X_n - n)/\sqrt{n}$?

(3) Let U, V be independent $N(0,1)$ r.v.'s.

(a) Given $\mu_X, \mu_Y \in \mathbb{R}$, $\sigma_X, \sigma_Y > 0$ and $\rho \in [-1,1]$, find a, b, c such that letting

$$(*) \qquad X = \mu_X + aU, \qquad Y = \mu_Y + bU + cV,$$

(X, Y) is multivariate normal with covariance matrix

$$\begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}.$$

(b) Find the conditional density of Y given $X = x$, expressed in terms of $\mu_X, \mu_Y, \sigma_X, \sigma_Y$, and ρ. HINT: Use $(*)$ directly.

(c) Let $\xi_1, \xi_2, \ldots$ be iid normal $N(0,1)$ random variables and $S_n = \sum_{k=1}^n \xi_i$. For $m < n$, find the conditional density of S_m given $S_n = c$, for all c.