Numerical Analysis Preliminary Examination
Spring 2023

January 8, 2023

Problem 1. (20 points)

Suppose $A \in \mathbb{R}^{n\times n}$ is symmetric and let $\|\cdot\|_F$ denote the Frobenius norm on $\mathbb{R}^n$ defined by

$$\|B\|_F^2 = \sum_{i,j=1}^{n} B_{ij}^2.$$

Given two integers $1 \le p < q \le n$ and an angle θ, let $c = \cos\theta$ and $s = \sin\theta$ and consider the matrix $G = G(p,q,\theta)$ for which

$$\begin{aligned} &G_{kk} = 1 \quad k \neq p,q \qquad && G_{ij} = 0 \quad i \neq j \text{ and } (i,j) \neq (p,q) \text{ or } (q,p) \\ &G_{pp} = c && G_{pq} = -s \\ &G_{qq} = c && G_{qp} = s \end{aligned}$$

This matrix is shown below.

$$G = G(p,q,\theta) = \begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & \ddots & & & & & & \\ & & & c & \dots & -s & & & \\ & & & \vdots & \ddots & \vdots & & & \\ & & & s & \dots & c & & & \\ & & & & & & \ddots & & \\ & & & & & & & 1 & \\ & & & & & & & & 1 \end{bmatrix}$$

Consider the matrix $B = G^T A G$. Notice, if $i,j \neq p,q$ then $B_{ij} = A_{ij}$, if $i \neq p,q$ then

$$\begin{aligned} B_{ip} &= cA_{ip} + sA_{iq} \\ B_{iq} &= -sA_{ip} + cA_{iq}, \end{aligned}$$

if $j \neq p, q$ then

$$B_{pj} = cA_{pj} + sA_{qj}$$
$$B_{qj} = -sA_{pj} + cA_{qj},$$

and

$$B_{pp} = c^2 A_{pp} + 2csA_{pq} + s^2 A_{qq}$$
$$B_{pq} = B_{qp} = (c^2 - s^2)A_{pq} + cs(-A_{pp} + A_{qq})$$
$$B_{qq} = c^2 A_{qq} - 2csA_{pq} + s^2 A_{pp}.$$

Jacobi's iterative method for finding the eigenvalues of A starts with the matrix $A^{(0)} = A$. For $k = 1, 2, \ldots$, values of $p_k < q_k$ and θ_k are chosen and the matrix $A^{(k)} = G^T A^{(k-1)} G$ is constructed.

(a) (5 points) Show that, for all k, $A^{(k)}$ is symmetric, $\|A^{(k)}\|_F^2 = \|A\|_F^2$, and the eigenvalues of $A^{(k)}$ are the same as the eigenvalues of A.

(b) (5 points) Given values of p_k and q_k, find a value of θ_k so that $A^{(k)}_{p_k q_k} = 0$.

(c) (5 points) Let $A^{(k)} = D^{(k)} + E^{(k)}$ where $D^{(k)}$ is diagonal and $E^{(k)}$ has zeros on the diagonal. If θ_k is chosen as in part (b) then it is a straightforward calculation to see that

$$||E^{(k)}||_F^2 = ||E^{(k-1)}||_F^2 - 2\left(A^{(k-1)}_{p_k q_k}\right)^2.$$

(You do not need to show this.) Show that $p_k < q_k$ can be chosen at each stage so that

$$\|E^{(k)}\|_F^2 \leq \left(1 - \frac{2}{n^2}\right)^k \|E^{(0)}\|_F^2.$$

(d) (5 points) Let $\varepsilon > 0$ be given. Show there exists K such that for all $k \geq K$, every eigenvalue of A lies within ε of a diagonal element of $A^{(k)}$ and every diagonal element of $A^{(k)}$ lies within ε of an eigenvalue of A.

Problem 2. (20 points)

(*a*) (5 points) Suppose $B \in \mathbb{R}^{n\times n}$ and consider the sequence defined by

$$x^{(k+1)} = Bx^{(k)} + d$$

where $d \in \mathbb{R}^n$. Show the sequence converges for all d and all $x^{(1)}$ to a limit that is independent of $x^{(1)}$ if and only if $B^k \to 0$. *Hint. Show that the matrix $I - B$ is invertible if $B^k \to 0$.*

(*b*) (5 points) Suppose $B \in \mathbb{R}^{n\times n}$. Show that B^k converges to zero as k goes to infinity is equivalent to the spectral radius of B being less than 1. *Hint. Write $SBS^{-1} = J$, where J is the Jordan form of B:*

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_r \end{bmatrix} \qquad \text{where} \qquad J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda i & 1 & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}.$$

Also, write $J_i = \lambda_i I + N$ where N is a nilpotent matrix and use the binomial theorem on J_i.

(*c*) (5 points) Let $Ax = b$, where $A \in \mathbb{R}^{n\times n}$ is a non singular matrix. Use (b) to show that if A is strictly row diagonally dominant then the Jacobi method converges for any arbitrary choice of the initial value $x^{(1)}$. *Hint: Recall that, for any induced norm, $\rho(B) \leq \|B\|$ where $\rho(B)$ denotes the spectral radius of B.*

(*d*) (5 points) Let

$$A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix}.$$

Apply two iterations of the Jacobi method to the system $Ax = b$ starting at $x^{(1)} = (0, 0, 0)^T$.

Problem 3. (20 points)

For $n = 1, 2, ...,$ let $\{\varphi_j^n\}_{j=0}^n \subset C[0,1]$ be given by $\varphi_0^n(x) = 1 - nx$ if $x \in [0, \frac{1}{n}]$, $\varphi_0^n(x) = 0$ otherwise, $\varphi_n^n(x) = nx - n + 1$ if $x \in [\frac{n-1}{n}, 1]$, $\varphi_n^n(x) = 0$ otherwise, and for $j = 1, 2, ..., n-1$, $\varphi_j^n(x) = nx - j + 1$ if $x \in [\frac{j-1}{n}, \frac{j}{n}]$, $\varphi_j^n(x) = j + 1 - nx$ if $x \in [\frac{j}{n}, \frac{j+1}{n}]$, $\varphi_j^n(x) = 0$ otherwise.

(a) (2 points) Sketch the graphs of φ_j^n for $j = 0, 1, 2, ..., n$

(b) (4 points) Compute the Gramian matrix, M^n, considered as vectors in $L_2(0,1)$, $M_{i,j}^n = \langle \varphi_i^n, \varphi_j^n \rangle_{L_2(0,1)}$ of $\{\varphi_j^n\}_{j=0}^n$.

(c) (3 points) Let $S = \text{span}\{\varphi_j^n\}_{j=0}^n$ and use part (b) to argue that $\{\varphi_j^n\}_{j=0}^n$ is in fact a basis for the subspace $S \subset L_2(0,1)$.

(d) (3 points) Let $\varphi \in C[0,1]$ be given and find an expression for $I^n\varphi \in S$, where I^n denotes the interpolation operator on $C[0,1]$ with respect to the mesh $\{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1\}$ on $[0,1]$; that is, $(I^n\varphi)(\frac{j}{n}) = \varphi(\frac{j}{n})$, $j = 0, 1, 2, ..., n$.

(e) (3 points) Let $\varphi \in L_2(0,1)$ be given and let P^n be the orthogonal projection of $L_2(0,1)$ onto S. Then $P^n\varphi = \sum_{j=0}^n b_j\varphi_j^n \in S$, for some $\mathbf{b} = [b_0, b_1, ..., b_n]^T \in \mathbb{R}^{n+1}$. What is (i.e. compute or provide an expression for) $\mathbf{b}$?

(f) (2 points) Given that for $\varphi \in C^2(0,1)$, $\|I^n\varphi - \varphi\|_{L_2(0,1)} \le \frac{K_0}{n^2}\|D^2\varphi\|_{L_2(0,1)}$ for some constant K_0, argue that $\|P^n\varphi - \varphi\|_{L_2(0,1)}$ is also $O(\frac{1}{n^2})$ for $\varphi \in C^2(0,1)$.

(g) (3 points) Use (1) the fact that for $\varphi \in C^2(0,1)$, $\|DI^n\varphi - D\varphi\|_{L_2(0,1)} \le \frac{K_0}{n}\|D^2\varphi\|_{L_2(0,1)}$, (2) Part (f) above, and (3) the Schmidt inequality which states that

$$\int_a^b |Dp_k(x)|^2 dx \le \frac{C_k}{(b-a)^2}\int_a^b |p_k(x)|^2 dx, \quad k = 1, 2, 3, or$$

where p_k is a polynomial of degree k and C_k is a constant that depends only on k, and not on a, b or p_k to argue that $\|DP^n\varphi - D\varphi\|_{L_2(0,1)}$ is also $O(\frac{1}{n})$ for $\varphi \in C^2(0,1)$. Note: D denotes the differentiation operator, $D = \frac{d}{dx}$. (Hint: Use the fact that $\int_0^1 = \sum_{j=1}^n \int_{\frac{j-1}{n}}^{\frac{j}{n}}$ and the triangle inequality and consider $\|DP^n\varphi - DI^n\varphi\|_{L_2(0,1)}$.)