DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Fall 2012

1. Consider the product space $\mathbb{R}^2 \times \mathbb{R}$ and let $P : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ be projection onto the second factor, $P(x,t) = t$. For two t values, $t_1 < t_2$ define the "copies" of $\mathbb{R}^2$, $X = P^{-1}(t_1)$ and $Y = P^{-1}(t_2)$ and the mapping

$$T : X \to Y \quad \text{where} \quad y = T(x) = \phi(t_2, t_1, x)$$

where $\phi(t_2, t_1, x)$ is the solution $\phi(t, t_1, x)$ of $y' = A(t)y, \quad y(t_1) = x$, evaluated at t_2, and

$$A(t) = \begin{pmatrix} -2 + \sin t & \cos t \\ \sin 4t & 1 + \cos t \end{pmatrix}.$$

Show (1) T is a diffeomorphism and (2) if μ is Borel measure then

$$\mu(T(B)) \to 0 \quad \text{as} \quad t_2 \to 0,$$

for any Borel set B.

2. Define the mapping $f : \mathbb{R}_+ \to \mathbb{R}_+ = [0, \infty)$ by

$$f(x) = \frac{ax^2}{1 + x^2}, \qquad a > 0,\ x \geq 0.$$

As a increases, starting near zero, a bifurcation takes place. Describe the type of bifurcation, find the critical value a_{c} and the point x_{bif} at which the bifurcation takes place.

3. Consider the mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x) = \begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix} x + \begin{pmatrix} x_1^2 \\ 2x_2^2 \end{pmatrix}.$$

Show

(i) The first quadrant $x_1 \geq 0,\ x_2 \geq 0$, call it Q_+, is invariant, i.e.$f(Q_+) \subset Q_+$ and

(ii) $x \in Q_+,\ x \neq (0,0), \implies \|f^n(x)\| \to \infty$ as $n \to \infty$, where

$$f^n \doteq f \circ f \circ \cdots \circ f.$$

4. Let u be a smooth solution of

$$\begin{aligned} &u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^3 \times (0, \infty) \\ &u = g, u_t = h \text{ on } \mathbb{R}^3 \times \{t = 0\}, \end{aligned}$$

where g and h are smooth and have compact support. Prove the existence of $C > 0$ such that

$$|u(x,t)| \leq \frac{C}{t}$$

for all $x \in \mathbb{R}^3$ and $t > 0$.

5. (a) Solve the linear partial differential equation

$$e^x u_x + u_y = u \text{ with } u(x,0) = g(x).$$

(b) Solve the nonlinear partial differential equation

$$x^2 u_x + y^2 u_y = u^2 \text{ with } u = 1 \text{ on the line } y = 2x$$

6. Write down the explicit formula for the solution of

$$\begin{aligned} u_t - \Delta u + cu &= f \text{ in } \mathbb{R}^n \times (0,\infty) \\ u &= g \text{ on } \mathbb{R}^n \times \{t=0\} \end{aligned}$$

DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Fall 2013

1. Show that $u \equiv 0$ is an asymptotically stable solution of
$$u'' + (1 - u^2)u' - (u')^3 + u^5 = 0.$$

2. Consider the matrix
$$\Phi(t) = \begin{bmatrix} t^2+1 & 0 & 0 & 3 \\ 4 & t-1 & 2 & t \\ 1 & 2 & t^3+2 & 2 \\ 3 & 1 & 1 & t \end{bmatrix}.$$
on the interval $t \in (a, b)$.

   (i) Give conditions on a and b so that Φ can be a fundamental matrix for a system
   $$x' = A(t)x, \qquad x \in \mathbb{R}^4,$$
   on the entire interval $t \in (a, b)$.

   (ii) Can $(a, b) = (-\infty, \infty)$?

3. Let $f \in C^1(R^n, R^n)$ and assume $V \in C^1(\mathbb{R}^n, \mathbb{R})$ is non-increasing along orbits of
$$x' = f(x), \qquad x(0) = x_0, \tag{1}$$
i.e. if $\phi(t, x_0)$ is the solution of (1) then
$$V(\phi(t_1, x_0)) \leq V(\phi(t_2, x_0)),$$
whenever $t_1 \geq t_2$. Suppose for some x_0, the ω-limit set, $\omega(x_0)$ is nonempty and bounded. Prove the restriction $V|\omega(x_0) =$ constant.

4. Let $U = \{x \in \mathbb{R}^n : |x| > 1\}$, Suppose $u \in C^2(U) \cap C(\bar{U})$ is a bounded solution of the following Dirichlet problem: $\Delta u = 0$ in U and $u = \varphi$ on $\Gamma = \{x \in \mathbb{R}^n : |x| = 1\}$, with $\varphi \in C(\Gamma)$.

   a) If $n = 2$, show that there exists at most one solution of the above problem

   b) If n=3, show that it is possible to have more than one bounded solutions of the above problem. What additional condition should you impose so that the solution is unique?

5. Find the explicit solution $u = u(t, x_1, x_2)$ of
$$\partial_t u + x_2 \partial_{x_1} u - x_1 \partial_{x_2} u = 0, \quad t > 0$$
subject to $u(t = 0, x_1, x_2) = e^{-x_1^2}$.

6. Let $u(x, t)$ solve the wave equation $u_{tt} - c^2 u_{xx} = q(x, t)$ for $x \in \mathbb{R}, \ t > 0$. with the initial condition $u(x, 0) = 0$ and $u_t(x, 0) = 0$. Here c is positive and $q(x, t) = (1 - x^2)\sin t$ for $|x| \leq 1$ and $q(x, t) = 0$ for $|x| > 1$. SHow that $u(x, t) = 0$ for $|x| > ct + 1$.

7. Let $u \in W^{1,p}(\mathbb{R}^n)$ where $p > n$. Show that
$$\frac{1}{r^n} \int_{B(x,r)} |u(x) - u(y)| \, dy \leq C(n) \int_{B(x,r)} \frac{|Du(y)|}{|x - y|^n} \, dy$$
where $Du(y)$ is the gradient of u ($Du(y) = (\partial_1 u(y), \ ...., \partial_n u(y))$, $B(x, r)$ is the euclidean ball with center x and radius r, and $C(n)$ is a constant depending only on n.

DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Fall 2015

The exam consists of seven problems. The first three problems are related to ODEs while the last four are related to PDEs. Good luck!

1. Consider the following system of ODE's

$$\begin{aligned} x' &= x + 2y - x^2 - y^2 + x^3 y - x^3 \\ y' &= 3x + 3y + xy^2 - xy \end{aligned}$$

   Let C be a small circle centered at the origin. What is the measure of the set of initial conditions on C that give rise to solutions $(x(t), y(t))$ with the property that

$$\lim_{t\to\infty} (x(t), y(t)) = (0, 0)?$$

   Assume $\mu(C) = 1$.

2. For any $x_0 \in \mathbb{R}$ show that the solution of

$$\frac{dx}{dt} = \frac{1 + x^2}{2 + x + 3x^2}, \qquad x(0) = x_0$$

   can be continued to all $t \in \mathbb{R}$.

3. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function with $f(0) \neq 0$ and A an $n \times n$ matrix with $\det A \neq 0$. Consider the system of differential equations

$$\frac{dx}{dt} = Ax + \mu f(x), \qquad \text{ODE}$$

   where $\mu \in [0, \mu_0)$ is a small parameter. Show that for $\mu_0 > 0$ sufficiently small there exists a C^1 family of stationary points $\xi(\mu)$ of ODE such that $\xi(0) = 0$.

4. Let $\Omega = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 1\}$ and assume that $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function with $f(0) = 0$. Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be a solution of the following problem

$$\Delta u = f(u) \quad (x, y) \in \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

   Show that u is identically zero.

5. Prove that for any function $f \in H^1(0, \pi)$ the following inequality holds

$$\int_0^\pi f^2\, dx \leq \int_0^\pi (f')^2\, dx + \left(\int_0^\pi f\, dx \right)^2.$$

6. Consider the 3-dimensional wave equation on a half space

$$u_{tt} = \Delta u \qquad x \in \Omega = \{(x_1, x_2, x_3);\ x_1 > 0\},$$

with the initial and boundary conditions

$$u(0,x) = g(x), \quad u_t(0,x) = 0; \qquad u(t,x) = 0 \text{ for } x \in \partial\Omega.$$

Assume that g is a smooth function with bounded support, which vanishes on $\partial\Omega = \{(x_1, x_2, x_3);\ x_1 = 0\}$,

(i) Write an explicit formula for the solution $u = u(x,t)$.

(ii) Prove that $\lim_{t\to\infty} ||u(t,\cdot)||_{L^\infty} = 0$.

(iii) Prove that the total energy

$$E = \frac{1}{2}\int_\Omega \left(u_t^2 + |\nabla u|^2\right) dx$$

is constant in time.

7. Write a brief discussion of the method of characteristics and then demonstrate the method by solving

$$u_x u_y = u \quad \text{in } \Omega \qquad\qquad u = y^2 \text{ on } \partial\Omega.$$

where $\Omega = \{(x,y)\ ;\ x > 0\} \subset \mathbb{R}^2$.

DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Spring 2015

1. Consider the initial value problem for the one dimensional wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0 \quad \text{in } \mathbb{R} \times (0, \infty) \\ u = g, \; u_t = h \quad \text{on } \mathbb{R} \times \{t = 0\} \end{cases}$$

Prove that if $g \in C^k$ and $h \in C^{k-1}$, then $u \in C^k$, but not in general smoother.

2. Consider the Euler equations for the flow of a constant density, inviscid fluid in three dimensions:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla P; \qquad \nabla \cdot u = 0,$$

where the vector $u(x,t)$ is the fluid velocity and the scalar $P(x,t)$ is the pressure.

Let $(\overline{u}(x), \overline{P}(x))$ be a stationary solution of the system (steady flow). Prove that the quantity

$$H = \overline{P} + \frac{1}{2}\overline{u}^2$$

is constant along the curves $dx/dt = \overline{u}(x)$.

3. Let $f(r) = r^5 - 4r^3$. Show that if u is a solution of the problem

$$\begin{cases} \Delta u = f(u) \quad \text{in } U \\ u = 0 \quad \text{on } \partial U \end{cases}$$

where U is a bounded open subset of $\mathbb{R}^n$, then necessarily $|u(x)| < 2$ throughout U.

4. Show that all solutions of

$$u'' + 4u + 3u^5 = 0$$

are bounded. If $u(0) = 1$ and $u'(0) = 2$ give a sharp bound on $u'(t)$, i.e. the least upper bound for $|u'(t)|$.

5. Consider the system of ODE's,

$$\begin{aligned} x_1' &= -x_1 - x_1 x_2^2, \\ x_2' &= \mu x_2 - 2x_3 - x_2(x_2^2 + x_3^2), \\ x_3' &= 2x_2 + \mu x_3 - x_3(x_2^2 + x_3^2). \end{aligned}$$

Ler S be the set of initial conditions close to the origin whose solutions approach the origin as $t \to \infty$. (1) For $\mu = 1$ determine the dimension of S. (2) Describe the qualitative change that takes place when μ is varied continuously from -1 to $+1$.

6. Consider the periodic system with period T,

$$x' = A(t)x, \qquad A(t + T) = A(t).$$

Show there is a solution $\phi(t)$ and a constant λ such that $\phi(T) = \lambda\phi(0)$.

DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Spring 2016

Each problem is worth 10 points. There are seven problems.

1. Find the stationary points and analyze their behavior as r varies from -1 to 1. Also determine the stability of each fixed point for each r.

$$\begin{aligned} x' &= rx - x^3 - y^2 - axy^3 \\ y' &= -y - x^2y^3 \end{aligned}$$

If there is a bifurcation then draw a bifurcation diagram.

2. Given the ODE
$$\frac{d^2x}{dt^2} + x + x^3 = 0,$$
choose the correct answer and prove it is correct:
   - There exists a solution that $\to 0$ as $t \to \infty$.
   - All non identically zero solutions are bounded and bounded away from zero,
   - There exists a solution that $\to \infty$ as $t \to \infty$.

3. Show that the planar system

$$\begin{aligned} x' &= xy^2 + y^4 + x^3 \\ y' &= x^2y + 2y^5, \end{aligned}$$

can have no periodic orbits in $\mathbb{R}^2$ other than the trivial one at $(0,0)$.

4. Let
$$u(x) = \log\log\left(1 + \frac{1}{|x|}\right).$$
Show that u is unbounded and that it is an element of the Sobolev space $H^1(B(0,1))$, where $B(0,1)$ is the unit ball in $\mathbb{R}^2$.

5. Let Ω be a bounded open set of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ and let $f : \mathbb{R} \mapsto \mathbb{R}$ be a function such that its derivative is bounded ($|f'| \le K$) and $f(0) = 0$. Assume that u is a C^2 solution of

$$\begin{aligned} \partial_t u - \Delta u = f(u) \qquad & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0 \qquad & \text{on } \partial\Omega \times (0,\infty). \end{aligned}$$

(i) Show that if $u(x,0) \ge 0$ for all $x \in \Omega$ then $u(x,t) \ge 0$ for all $x \in \Omega$ and all $t > 0$.

(ii) Show that if $u(x,0) \le M$ for all $x \in \Omega$ then $u(x,t) \le Me^{Kt}$ for all $x \in \Omega$ and all $t > 0$.

6. Consider the initial value problem

$$u_t + uu_x + u = 0, \qquad -\infty < x < \infty,\ t > 0 \tag{B}$$

subject to the initial condition $u(x,0) = a\sin x$.

(i) Find the characteristic curves associated with the equation (B) in an explicit form.

(ii) Show that if $a > 1$, then there exists a time $t = t(a)$ such that there exists no smooth solution (C^1in both space and time) of the equation (B) for $t > t(a)$. Find the maximal time of smoothness, $t(a)$.

7. Find the solution in the first quadrant $x > 0$ and $t > 0$ of the Wave equation with boundary condition at $x = 0$,

$$\begin{aligned} u_{tt} - c^2 u_{xx} = 0, \quad x > 0, \quad t > 0,\\ u(0,x) = f(x), \quad u_t(0,x) = g(x),\\ u_t(t,0) = au_x(t,0), \quad a \neq -c, \end{aligned}$$

where $f(x)$ and $g(x)$ are C^2 functions which vanish near $x = 0$. Show that no solution exists in general if $a = -c$.

DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Spring 2017

Each problem is worth 10 points. There are seven problems.

1. Consider the 2[nd] order ODE,

$$x'' + p(t)x' + a\,x = 0$$

where $a > 0$ and $\displaystyle\int_0^t p(s)ds \to \infty$ an $t \to \infty$. Suppose $\phi(t)$ and $\psi(t)$ form a fundamental set of solutions, i.e.

$$X(t) = \begin{pmatrix} \phi(t) & \psi(t) \\ \phi'(t) & \psi'(t) \end{pmatrix}$$

is non-singular. Prove $\det X(t) \to 0$ as $t \to \infty$.

2. Given the system of ODE's

$$x' = Ax + f(t,x),$$

where A is an $n \times n$ real matrix with spectrum in the open left half of the complex plane. The function f is real, continuous for small $|x|$ and $t \geq 0$ and

$$f(t,x) = o(|x|), \qquad \text{as} \quad |x| \to 0,$$

uniformly in t, i.e. given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\frac{|f(t,x)|}{|x|} \leq \varepsilon,$$

whenever $0 < |x| < \delta$, and $t \in [0,\infty)$. Prove that the zero solution is asymptotically stable.

3. Show that the planar system

$$\begin{aligned} x' &= x - y - x^3 - xy^2 \\ y' &= x + y - x^2y - y^3, \end{aligned}$$

has a periodic orbit Π in $\mathbb{R}^2$ other than the trivial one at $(0,0)$. Is $(0,0)$ necessarily contained in the region surrounded by Π? Justify your answer.

4. Consider the initial value problem

$$\begin{aligned} &u_t + uu_x = 2, \qquad x \in \mathbb{R},\ t > 0, \\ &u(x,0) = x, \qquad\ \ x \in \mathbb{R}. \end{aligned}$$

   (a) Find the equation for the characteristics. Are there any shock forming with this initial condition?

   (b) Find an explicit formula for the solution of this initial value problem.

5. Let $U \subset \mathbb{R}^n$ be open and bounded and let $u \in C^2(U) \cap C(\bar{U})$ be harmonic in U. Prove that if U is connected and there exists $x_0 \in U$ such that $u(x_0) = \max_{\bar{U}} u$, then u is constant within U.

6. Let U be a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with a smooth boundary and let u be a C^2-solution of the following initial boundary value problem

$$\begin{aligned} &u_t = \Delta u \quad \text{in } U \times (0, \infty), \\ &u = 0 \quad \text{on } \partial U \times (0, \infty), \\ &u = u_0 \quad \text{on } U \times \{t = 0\}. \end{aligned}$$

Here u_0 is smooth, bounded and integrable in U. Show that there exists a constant $\alpha > 0$ such that

$$\|u(\cdot, t)\|_{L^2(U)} \leq e^{-\alpha t} \|u_0\|_{L^2(U)}.$$

7. We define a weak solution of the one-dimensional wave equation $u_{tt} - u_{xx} = 0$ to be a function $u(x, t)$ such that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, t) \left(\phi_{tt}(x, t) - \phi_{xx}(x, t) \right) dx dt = 0$$

for every $\phi \in C_c^2(\mathbb{R}^2)$.

(a) Show that any C^2-solution of the wave equation is also a weak solution.

(b) Determine if $u(x, t) := H(x - t)$ is a weak solution of the wave equation or not. Here H is the Heaviside function: $H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x \geq 0$.

PARTIAL DIFFERENTIAL EQUATIONS QUALIFYING EXAM
Spring 2022

Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Start each problem on a fresh sheet of paper and write on only one side of the paper.

1. Consider a solution u to the nonlinear equation

$$\begin{cases} -\Delta u = \lambda u^2(1-u) & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\lambda > 0$ is a constant and Ω is a bounded domain with smooth boundary. Prove that $0 \le u \le 1$ in Ω.

2. Let $u(x,t)$ be a smooth solution of the initial value problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u(x,0) = g(x),\ u_t(x,0) = h(x) & \text{for } x \in \mathbb{R}^3, \end{cases}$$

where $g, h \in C^\infty(\mathbb{R}^3)$ and are supported in the ball $B(0,R)$.

(a) Use the Kirchoff formula to show that

$$|t \cdot u(x,t)| \le C\left(2 + \frac{1}{t}\right) |\partial B(x,t) \cap B(0,R)|$$

for all $(x,t) \in \mathbb{R}^3 \times (0,\infty)$, where $|\cdot|$ denotes two-dimensional Lebesgue measure.

(b) Show that for some $t_0 > 0$ the measure $|\partial B(x,t) \cap B(0,R)|$ can be bounded by a constant independent of $t \ge t_0$, and conclude that

$$|u(x,t)| \le \frac{C}{t}$$

for all $(x,t) \in \mathbb{R}^3 \times (0,+\infty)$, where $C > 0$ is a constant.

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and assume that $u(x,t)$ is a nonnegative function in $C^2(\overline{\Omega} \times [0,+\infty))$, which solves the heat conduction equation with heat loss due to radiation

$$\begin{cases} (\partial_t - \Delta)u = -u^4 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

prove that we can find a constant C *independent of* the initial data $u(0)$, such that

$$E(1) := \int_\Omega u(x,1)^2 \, \mathrm{d}x \le C.$$

ODE EXAM - Fall 2022

The exam has **four** problems on **two** pages. Each problem is worth 10 points. Do all four problems.

1. Let $a : [0, \infty) \to [0, \infty)$ and $u : [0, \infty) \to [0, \infty)$ be two nonnegative continuous functions. Assume that

$$u(x) \le \int_0^x a(y)u(y)dy$$

for all $x \ge 0$. Show, *without citing* Gronwal's inequality, that $u(x) = 0$ for $x \ge 0$. To clarify, you cannot simply claim that the result follows from Gronwal's inequality; instead, you either *establish* Gronwal's inequality in this setting or use some other argument.

2. Consider the 2^{nd} order ODE for the unknown function $x = x(t)$,

$$x'' + p(t)x' + ax = 0,$$

where $p(t) = 2 - 3\cos(t)$ and a is a real number. Suppose $\phi(t)$ and $\psi(t)$ form a fundamental set of solutions, i.e. the matrix

$$X(t) = \begin{pmatrix} \phi(t) & \psi(t) \\ \phi'(t) & \psi'(t) \end{pmatrix}$$

is non-singular. Prove that

$$\lim_{t \to +\infty} \det X(t) = 0,$$

that is, the determinant of the matrix $X(t)$ converges to zero as $t \to +\infty$.

3. Consider the system for the unknown functions $x(t)$ and $y(t)$:

$$\begin{cases} x' = -y - y^2x^3 \\ y' = x - x^2y. \end{cases} \tag{1}$$

(i) Identify the stationary points of the system.

(ii) Prove that the system (1) has a unique solution $(x(t), y(t))$ satisfying $x(0) = 1,\ y(0) = 0$, and the solution is defined for all $t \ge 0$.

(iii) Prove that the solution from part (ii) satisfies

$$\lim_{t \to +\infty} x(t) = \lim_{t \to +\infty} y(t) = 0.$$

4. Consider the two-dimensional ODE

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}), \tag{2}$$

where the vector field $\boldsymbol{f}$ is continuously differentiable everywhere in $\mathbb{R}^2$. Suppose Γ is a periodic orbit for (2).

(i) What can we conclude about the index of Γ with respect to $\boldsymbol{f}$? Give a short explanation.

(ii) What can we conclude about the number and type of stationary points of $\boldsymbol{f}$ inside the region enclosed by Γ? Provide as many details as you can.

(iii) Which of the following statements about the stationary points of $\boldsymbol{f}$ inside the region enclosed by Γ are definitely NOT true? Explain your conclusions.

1. $\boldsymbol{f}$ has exactly one stationary point inside the region enclosed by Γ, and the point is a saddle.
2. $\boldsymbol{f}$ has exactly two stationary points inside the region enclosed by Γ: a saddle and a center.
3. $\boldsymbol{f}$ has exactly three stationary points inside the region enclosed by Γ: a saddle, a center, and a stable node.

ODE EXAM - Spring 2023

The exam has **four** problems on **two** pages. Each problem is worth 10 points. Do all four problems.

Your work should be neat and well organized. Neatness will not be officially taken into account in the scoring, but a greater degree of clarity will allow the committee to more confidently evaluate your work.

1. Let $a : [0, \infty) \to [0, \infty)$ and $u : [0, \infty) \to [0, \infty)$ be two nonnegative continuous functions. Assume that

$$u(x) \leq \int_0^x a(y)u(y)dy$$

for all $x \geq 0$. Show, *without citing* Gronwal's inequality, that $u(x) = 0$ for $x \geq 0$. To clarify, you cannot simply claim that the result follows from Gronwal's inequality; instead, you either *establish* Gronwal's inequality in this setting or use some other argument.

2. Consider the 2^{nd} order ODE for the unknown function $x = x(t)$,

$$x'' + p(t)x' + ax = 0,$$

where $p(t) = 2 - 3\cos(t)$ and a is a real number. Suppose $\phi(t)$ and $\psi(t)$ form a fundamental set of solutions, i.e. the matrix

$$X(t) = \begin{pmatrix} \phi(t) & \psi(t) \\ \phi'(t) & \psi'(t) \end{pmatrix}$$

is non-singular. Prove that

$$\lim_{t \to +\infty} \det X(t) = 0,$$

that is, the determinant of the matrix $X(t)$ converges to zero as $t \to +\infty$.

3. Consider the system for the unknown functions $x(t)$ and $y(t)$:

$$\begin{cases} x' = -y - y^2x^3 \\ y' = x - x^2y. \end{cases} \tag{1}$$

(i) Identify the stationary points of the system.

(ii) Prove that the system (1) has a unique solution $(x(t), y(t))$ satisfying $x(0) = 1,\ y(0) = 0$, and the solution is defined for all $t \geq 0$.

(iii) Prove that the solution from part (ii) satisfies

$$\lim_{t \to +\infty} x(t) = \lim_{t \to +\infty} y(t) = 0.$$

4. Consider the two-dimensional ODE

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}), \tag{2}$$

where the vector field $\boldsymbol{f}$ is continuously differentiable everywhere in $\mathbb{R}^2$. Suppose Γ is a periodic orbit for (2).

(i) What can we conclude about the index of Γ with respect to $\boldsymbol{f}$? Give a short explanation.

(ii) What can we conclude about the number and type of stationary points of $\boldsymbol{f}$ inside the region enclosed by Γ? Provide as many details as you can.

(iii) Which of the following statements about the stationary points of $\boldsymbol{f}$ inside the region enclosed by Γ are definitely NOT true? Explain your conclusions.

1. $\boldsymbol{f}$ has exactly one stationary point inside the region enclosed by Γ, and the point is a saddle.
2. $\boldsymbol{f}$ has exactly two stationary points inside the region enclosed by Γ: a saddle and a center.
3. $\boldsymbol{f}$ has exactly three stationary points inside the region enclosed by Γ: a saddle, a center, and a stable node.