Fall 2008 Math 541b Exam

1. Let $X_1, \ldots, X_n$ be i.i.d. from a normal distribution with unknown mean μ and known variance 1. Suppose that negative values of X_i are truncated at 0, so that instead of X_i we actually observe

$$Y_i = \max\{0, X_i\}, \quad i = 1, \ldots, n,$$

from which we would like to estimate μ.

   (a) Explain how to use the EM algorithm to estimate μ from $Y_1, \ldots, Y_n$. Specifically, give the complete log-likelihood function $\log L_c(\mu)$ (i.e., the log of the joint density of $X_1, \ldots, X_n$) and a recursive formula for the successive EM estimates $\mu^{(k+1)}$. Write these in terms of the density ϕ and c.d.f. Φ of the standard normal distribution. *Hint:* To simplify things, assume that $X_1, \ldots, X_m$ are not truncated, and $X_{m+1}, \ldots, X_n$ are.

   (b) Find the partial log-likelihood function $\log L(\mu)$ (i.e., the log of the joint density of $Y_1, \ldots, Y_n$) and use it to write down a (nonlinear) equation which the MLE $\widehat{\mu}$ satisfies. Use this equation to manually verify that $\widehat{\mu}$ is indeed a fixed point of the recursion found in (a).

2. Let f denote the true density function of X, and consider testing the simple hypotheses

$$H_0 : f = f_0 \quad \text{vs.} \quad H_1 : f = f_1$$

for given densities f_0, f_1. For a fixed value $\pi \in (0, 1)$, suppose that the probabilities $\pi_0 = \pi$ and $\pi_1 = 1 - \pi$ can be assigned to H_0 and H_1 prior to the experiment. We will describe tests of H_0 vs. H_1 by their indicator functions

$$\psi(X) = \begin{cases} 1, & \text{the test rejects } H_0 \\ 0, & \text{the test accepts } H_0. \end{cases}$$

   (a) Show that the overall probability of an error resulting from using a test ψ is

$$\pi E_0 \psi(X) + (1 - \pi) E_1[1 - \psi(X)]. \tag{1}$$

   (b) Call the test ψ^* minimizing (1) the *Bayes optimal* test. By writing (1) as a single E_0 expectation using the "change of measure" technique

$$E_1(\cdot) = E_0 \left[(\cdot) \frac{f_1(X)}{f_0(X)} \right],$$

   show that the Bayes optimal test is equivalent to a simple likelihood ratio test. Also, give the value of the likelihood ratio test's critical value.

   (c) Argue that the Bayes optimal test is hence most powerful for detecting f_1 at a certain significance level. Write down an expression for this significance level, and also give an upper bound for it as a function of π.

   (d) The *posterior probability* of H_i is the conditional probability that H_i is true, given $X = x$. Show that the posterior probability of H_i is

$$\frac{\pi_i f_i(x)}{\pi_0 f_0(x) + \pi_1 f_1(x)}. \tag{2}$$

   Show that the Bayes optimal test is also equivalent to choosing which hypothesis has the larger posterior probability.