Spring 2008 Math 541b Exam

1. Suppose that $X_1, \cdots, X_n$ are i.i.d. samples from the uniform distribution on $(0, \theta)$.

   (a) Show that the MLE of θ is
   $$\hat{\theta} = \max(X_1, \cdots, X_n)$$

   (b) Show that $n(\theta - \hat{\theta})$ converges in distribution to an exponential. Please specify the parameter.

   (c) Let $\lambda_n = \sup_\theta (L_n(\theta)/L_n(\theta_0))$, where $L_n(\theta)$ is the likelihood of $X_1, \cdots, X_n$. Show that
   $$2 \log \lambda_n = \begin{cases} 2n \log \frac{\theta_0}{\hat{\theta}} & \hat{\theta} \le \theta_0 \\ \infty & \hat{\theta} > \theta_0 \end{cases}$$

   (d) Show that $2 \log \lambda_n \longrightarrow \chi_2^2$ in distribution. Please notice that the degree of freedom is 2.

   (e) What is the asymptotic distribution of the likelihood ratio test under the general regularity conditions? Is the result in the last part consistent with the general result?

2. Consider the following formulation of the EM algorithm for the estimation of the parameter $\psi \in \Omega$ upon observing incomplete data y which is obtained through a (many to one) function as $y = y(x)$. The incomplete data is distributed according to $g(y; \psi)$; the full data x according to $g_c(x; \psi)$. These two densities are are related through the mapping $y = y(x)$ by
   $$g(y; \psi) = \int_{x:y=y(x)} g_c(x, \psi) dx.$$

   Begin with any initial value $\psi^{(0)} \in \Omega$, then iterate the following E and M steps.

   E step. Calculate
   $$Q(\psi, \psi^{(k)}) = E_{\psi^{(k)}} \left(\log L_c(\psi) | y \right).$$

   M step. Let $\psi^{(k+1)}$ be any value in Ω that maximizes $Q(\psi, \psi^{(k)})$, that is
   $$Q(\psi^{(k+1)}, \psi^{(k)}) \ge Q(\psi, \psi^{(k)}) \quad \text{for all } \psi \in \Omega.$$

(a) Argue that the conditional density of x given y when $y = y(x)$ is

$$k(x|y,\psi) = \frac{g_c(x;\psi)}{g(y;\psi)}, \tag{1}$$

and zero otherwise. Letting

$$H(\psi, \psi^{(k)}) = E_{\psi^{(k)}}(\log k(x|y,\psi)|y),$$

prove that

$$H(\psi, \psi^{(k)}) \le H(\psi^{(k)}, \psi^{(k)}) \quad \text{for all } \psi \in \Omega.$$

(b) Use (1) to write the log likelihood of the incomplete data $\log L(\psi) = \log g(y;\psi)$ as a difference involving the functions Q and H.

(c) Prove that the log likelihood sequence $\log L(\psi^{(k)})$ is monotone nondecreasing (Hint: consider differences).