Fall 2007 Math 541b Exam

1. Let $p_0(\mathbf{x})$ and $p_1(\mathbf{x})$ be two distinct density functions on $\mathbf{R}^d$.

   (a) Based on $\mathbf{X}_1, \ldots, \mathbf{X}_n$ independent and identically distributed with density p, it is desired to test $H_0 : p = p_0$ versus $H_1 : p = p_1$. Prove the Neymann Pearson Lemma, that the test which rejects H_0 when the likelihood ratio

   $$L_n = \prod_{i=1}^{n} \frac{p_1(\mathbf{X}_i)}{p_0(\mathbf{X}_i}$$

   exceeds a threshold has maximum power over all tests having the same Type I error.

   (b) Let $E_i, i \in \{0, 1\}$ be the expectation when the density of $\mathbf{X}$ is $p_i(\mathbf{x})$. Prove that

   $$K(1,0) < 0 \quad \text{where} \quad K(1,0) = E_0 \log \frac{p_1(\mathbf{X})}{p_0(\mathbf{X})}.$$

   (c) Let $\mathbf{X}_1, \mathbf{X}_2, \ldots$ be an infinite sequence of observations independent and identically distributed with density p, which are observed one at a time. Let $a < 0 < b$ and consider the test which, after n observations, rejects H_0 if $\log L_n > b$, accepts H_0 if $\log L_n < a$, and takes an additional observation otherwise. By considering $n^{-1} \log L_n$, show that this test always terminates, whether H_0 or H_1 is true.

2. Let $X_1, \ldots, X_n$ be i.i.d. with mean μ and variance σ^2, and g a function with continuous derivative.

   Show that the jackknife estimate of variance of

   $$\hat{\theta} = g(\overline{X})$$

   is asymptotically equivalent to what is produced using the delta method.