Spring 2008 Math 541a Exam

1. Let $X_1, \ldots, X_n$ be i.i.d. Poisson with mean λ.

   Note that
$$p_0 = P(X_i = 0) = e^{-\lambda},$$
   so if $Y = \#\{i : X_i = 0\}$, then λ might be estimated by
$$\widetilde{\lambda} = -\log(Y/n).$$

   (a) What is the distribution of Y?

   (b) Use Taylor series to find approximations of $E(\widetilde{\lambda})$ and $\mathrm{Var}(\widetilde{\lambda})$.

   (c) Compare (your approximation of) $\mathrm{Var}(\widetilde{\lambda})$ with the variance of the MLE $\widehat{\lambda}$ of λ.

   (d) Determine the Fisher information in the sample, and whether the MLE $\widehat{\lambda}$ achieves the Cramer-Rao lower bound.

2. Let $X_1, \cdots, X_n$ be i.i.d. $N(\mu, \sigma^2)$ and ϕ be the standard Gaussian density, and define the average absolute deviation, or AAD, by
$$\mathrm{AAD} = \frac{1}{n}\sum_{i=1}^{n} |X_i - \bar{X}|.$$

   (a) Show that
$$\sqrt{n}\{\mathrm{AAD} - 2\sigma\phi(0)\} \longrightarrow N(0, \sigma^2(1 - \frac{2}{\pi}))\,.$$

   (b) Let $\mathrm{med}(\mathbf{X})$ be the median of $\mathbf{X} = (X_1, \ldots, X_n)$, and $\mathrm{sign}(x) = -1, 0, 1$ if x is negative, zero or positive, respectively. Assuming
$$\mathrm{med}(\mathbf{X}) = \mu + \frac{1}{2n\phi(0)}\sum_{i=1}^{n} \mathrm{sign}(X_i - \mu) + o_p(n^{-1/2})\,,$$
   and using the bivarite CLT, deduce that
$$(\sqrt{n}(\mathrm{med}(\mathbf{X}) - \mu), \sqrt{n}(\mathrm{AAD} - \sigma\sqrt{\frac{2}{\pi}})) \longrightarrow N_2\left(0, \sigma^2\begin{bmatrix} \pi/2 & 0 \\ 0 & 1 - 2/\pi \end{bmatrix}\right)\,.$$

Hint: For part a, show that

$$\left|\frac{1}{n}\sum |X_i - \bar{X}| - \frac{1}{n}\sum |X_i - \mu|\right| = o_p(n^{-1/2})$$

by assuming (without loss of generality) that $\mu < \bar{X}$, verifying

$$\begin{aligned} & \left|\sum \{|X_i - \bar{X}| - |X_i - \mu|\}\right| \\ = \;& |\bar{X} - \mu|\left(\left|\sum\{\mathbf{1}(X_i \le \mu) - \mathbf{1}(X_i > \mu)\}\right| + 2\sum \mathbf{1}(\mu < X_i \le \bar{X})\right), \end{aligned}$$

and then applying the law of large numbers.
The following fact may also be useful: $E|X_1 - \mu| = 2\sigma\phi(0) = \sqrt{\frac{2}{\pi}}\sigma$.