# Fall 2007 Math 541a Exam

1. (a) Show that for all $\alpha > 0, \beta > 0$

$$G(x; \alpha, \beta) = e^{-\beta e^{-\alpha x}} \quad \text{for } x \in \mathbf{R}$$

is a distribution function (Gumbel family), and find its density and moment generating function.

(b) Is $G(x; \alpha, \beta)$ is a member of the exponential family? Find sufficient statistics for $X_1, \ldots, X_n$, an i.i.d. sample from $G(x; \alpha, \beta)$.

(c) Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with distribution function $F(x)$. Find the distribution function of

$$X_{(n)} = \max_{1 \le i \le n} X_i \quad \text{and} \quad X_{(1)} = \min_{1 \le i \le n} X_i.$$

(d) Let $G_1, \ldots, G_n$ be i.i.d. with distribution $G(x; \alpha, \beta)$ for some positive $\alpha$ and $\beta$. Show that the largest variable $G_{(n)} = \max_{1 \le i \le n} G_i$ is of the same family, and determine its parameters.

(e) Taking the distribution

$$F(x) = 1 - e^{-\lambda x} \quad \text{for } x \ge 0$$

in part a, find a sequence $b_n$ such that $X_{(n)} - b_n$ converges to the Gumbel distribution as $n \to \infty$, and determine the parameters of the limit distribution.

2. Let $(X_i, Y_i)$, $i = 1, \cdots, n$ be independent where $X_i$ has an exponential distribution $\mathcal{E}(\lambda_i)$ with density $p(x, \lambda_i) = \lambda_i e^{-\lambda_i x}$, $x > 0$, and $Y_i$ is independent of $X_i$ with the exponential distribution $\mathcal{E}(\theta \lambda_i)$, $\theta > 0$.

(a) Show that the maximum likelihood estimates of $(\lambda_1, \cdots, \lambda_n, \theta)$ are

$$\hat{\lambda}_i = \frac{2}{X_i + \hat{\theta} Y_i}, \quad , i = 1, \cdots, n$$

and $\hat{\theta}$, which uniquely solves $g(\theta) = 0$ where

$$g(\theta) = \frac{1}{n} \sum \left( \frac{X_i}{X_i + \theta Y_i} - \frac{1}{2} \right).$$

(b) Show that the Fisher information bound for $\theta$ under the assumption that $\lambda_1, \cdots, \lambda_n$ are known is $\theta^2/n$

(c) Show that $U_i = X_i/(X_i + \theta Y_i)$, $i = 1, \cdots, n$ are independent uniform $\mathcal{U}(0, 1)$.

(d) Now suppose $\hat{\theta} \xrightarrow{p} \theta$. Show that

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} \mathcal{N}(0, 3\theta^2)$$

by using the Taylor expansion

$$g(\hat{\theta}) = g(\theta) + g'(\theta)(\hat{\theta} - \theta) + o_p(\hat{\theta} - \theta)\,.$$