REAL ANALYSIS GRADUATE EXAM
Spring 2009

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1)(a) Let $(X, \mathcal{B}, \mu)$ be a measure space, with μ finite, and let $\mathcal{A} \subset \mathcal{B}$ be an algebra. A set $E \in \mathcal{B}$ is called *approximable from inside by* $\mathcal{A}$ if for every $\epsilon > 0$ there exists $A \in \mathcal{A}$ with $A \subset E, \mu(E \backslash A) < \epsilon$. Show that $\mathcal{C} = \{E \in \mathcal{B} : E$ is approximable from inside by $\mathcal{A}\}$ is closed under countable unions.

(b) Find an example which shows $\mathcal{C}$ need not be closed under complements. HINT: Consider the rationals and irrationals in an interval.

(2) Let f, g be absolutely continuous on $[a, b]$.

(a) Show that fg is absolutely continuous.

(b) Show that the integration by parts formula is valid:

$$\int_{[a,b]} fg' \, dx = f(b)g(b) - f(a)g(a) - \int_{[a,b]} f'g \, dx.$$

(c) Show by example that the integration by parts formula need not be valid if we only assume f, g differentiable a.e. (that is, we do not assume they are absolutely continuous.)

(3) Suppose $f : \mathbb{R} \to \mathbb{R}$ is integrable and $f = 0$ outside $[-1, 1]$. Define $f_n(x) = f(x + \frac{1}{n})$. Must $f_n \to f$ in measure? Justify your answer. HINT: What about convergence in L^1? Also, first consider a special subclass of the specified functions f.

(4) Let $\mu(X) < \infty$, and suppose $f \geq 0$ is measurable on X. Prove: f is μ-integrable iff

$$\sum_{n=0}^{\infty} 2^n \mu \left(\{x : f(x) \geq 2^n\} \right) < \infty.$$