Spring 2009 Math 541b Exam

1. Let $X_1, \ldots, X_{n_1}$ be i.i.d. $N(\mu_1, \sigma_1^2)$ and $Y_1, \ldots, Y_{n_2}$ i.i.d. $N(\mu_2, \sigma_2^2)$, where $\mu_1, \mu_2, \sigma_1, \sigma_2$ are all unknown and σ_1, σ_2 are not necessarily assumed to be equal. This problem concerns the generalized likelihood ratio (GLR) test of

$$H_0 : \mu_1 = \mu_2 \quad \text{vs.} \quad H_1 : \mu_1 \neq \mu_2.$$

   (a) Letting $L(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2)$ be the log of the likelihood function of $(X_1, \ldots, X_{n_1}, Y_1, \ldots, Y_{n_2})$, find the (unrestricted) maximum likelihood estimates (MLEs) $\widehat{\mu}_1, \widehat{\mu}_2, \widehat{\sigma}_1^2, \widehat{\sigma}_2^2$ and write down

$$L(\widehat{\mu}_1, \widehat{\mu}_2, \widehat{\sigma}_1^2, \widehat{\sigma}_2^2)$$

   in as simple form as possible.

   (b) For the H_0-restricted MLEs $\widehat{\widehat{\mu}}, \widehat{\widehat{\sigma}}_1^2, \widehat{\widehat{\sigma}}_2^2$,

      i. Write down formulas for $\widehat{\widehat{\sigma}}_i^2$ as functions of $\widehat{\sigma}_i^2$ and $\widehat{\widehat{\mu}}$,

      ii. Find a cubic equation, not depending explicitly on $\widehat{\widehat{\sigma}}_i$, that $\widehat{\widehat{\mu}}$ satisfies. You do not need to solve this equation.

   (c) Give the asymptotic distribution of the GLR statistic under H_0, as $n_1, n_2 \to \infty$.

2. Let $\widehat{\theta}_n$ be a parameter estimate computed from the random sample $X_1, \ldots, X_n$, let $\widehat{\theta}_{(i)}$ be the estimate computed from

$$X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n,$$

   and let

$$\widehat{\theta}_{(\cdot)} = \frac{1}{n} \sum_{i=1}^{n} \widehat{\theta}_{(i)}.$$

   Recall that the Jackknife estimate of the bias of $\widehat{\theta}$ is given by

$$b_{JACK} = (n-1)(\widehat{\theta}_{(\cdot)} - \widehat{\theta}_n).$$

   Now let $X_{(1)}, \ldots, X_{(n)}$ be the order statistics of the sample, which we assume are from a continuous distribution. The sample median $\widehat{m}_n$ is

$$\widehat{m}_n = \begin{cases} X_{((n+1)/2)}, & n \text{ odd} \\ (X_{(n/2)} + X_{((n/2)+1)})/2, & n \text{ even.} \end{cases}$$

   (a) Compute the jackknife estimate of bias b_{JACK} for the sample median $\widehat{m}_n$ in both cases.

   (b) Let b denote the true bias of $\widehat{m}_n$. Is b_{JACK} always unbiased for b in the even case?