Spring 2009 Math 541a Exam

1. Let $X_1, \ldots, X_n$ be a random sample from a Bernoulli distribution with parameter $p \in (0,1)$, that is,

$$P(X_i = 1) = p, \quad \text{and} \quad P(X_i = 0) = 1 - p.$$

   (a) Find a complete sufficient statistic $T_n(X_1, \ldots, X_n)$ for p.

   (b) Justify $T_n(X_1, \ldots, X_n)$ is sufficient and complete using the definitions of sufficiency and completeness.

   (c) Find the maximum likelihood estimator (MLE) of p, and determine is asymptotic distribution by a direct application of the Central Limit Theorem.

   (d) Calculate the Fisher information for the sample, and verify that your result in part (c) agrees with the general theorem which provides the asymptotic distribution of MLE's.

   (e) Find a variance stabilizing transformation for p, that is, a function g such that

$$\sqrt{n}(g(\hat{p}) - g(p)) \to_d \mathcal{N}(0, \sigma^2)$$

   where $\sigma^2 > 0$ and does not depend on p; identify both g and σ^2.

2. (a) Let θ have a Gamma $\Gamma(\alpha, \beta)$ distribution with α, β positive,

$$p(\theta; \alpha, \beta) = \frac{\theta^{\alpha-1} e^{-\theta/\beta}}{\beta^{\alpha} \Gamma(\alpha)},$$

   and suppose that the conditional distribution of X given θ is normal with mean zero and variance $1/\theta$.

   Show that the conditional distribution of θ given X also has a Gamma distribution, and determine its parameters.

   (b) Conditional on θ as in part (a), suppose that a sample $X_1, \ldots, X_n$ is composed of independent variables, normally distributed with mean zero and variance $1/\theta$. Find the conditional expectation $E[\theta | X_1, \ldots, X_n]$ of θ given the sample, and show that it is a consistent estimate of θ, that is, that

$$E[\theta | X_1, \ldots, X_n] \to_p \theta \quad \text{as } n \to \infty.$$