Geometry/Topology Qualifying Exam

Spring 2009

Solve all **SIX** *problems. Partial credit will be given to partial solutions.*

1. Let $S^2 = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$. Prove that the map
$$f : S^2 \to \mathbb{R}^4,\ f(x_1, x_2, x_3) = (x_1^2 - x_2^2, x_1x_2, x_1x_3, x_2x_3)$$
is an immersion and that $f(S^2)$ is diffeomorphic to the projective plane $\mathbb{RP}^2$.

2. Let ω be a closed n-form on $\mathbb{R}^{n+1} - \{0\}$. Prove that ω is exact if and only if $\int_{S^n} \omega = 0$, where S^n is the unit sphere in $\mathbb{R}^{n+1}$.

3. Find all vector fields Z on $\mathbb{R}^2$ which satisfy $[X, Z] = 0$ and $[Y, Z] = 0$, where $X = e^x \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y}$ are vector fields defined on all of $\mathbb{R}^2$.

4. Compute $\pi_n(T^p)$ for all $n \geq 1$, where $T^p = S^1 \times \cdots \times S^1$ (p times) is the p-dimensional torus.

5. Compute $\pi_1(\mathbb{R}^3 - K)$, where $K \subset \mathbb{R}^3$ is the union of the vertical axis $\{x = 0, y = 0\}$ and the unit circle $\{x^2 + y^2 = 1, z = 0\}$.

6. Let X be a compact, oriented surface of genus 2 (without boundary), and let A be a simple closed curve which separates the surface X into two punctured tori, as given in Figure 1 below. Then compute the relative homology groups $H_n(X, A)$ for all $n \geq 0$.

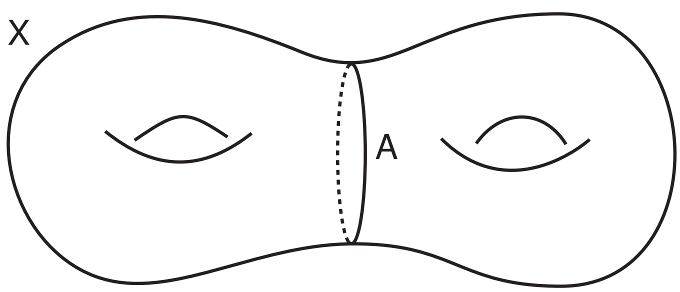

FIGURE 1