ALGEBRA QUALIFYING EXAM, Spring 2009

Throughout, $\mathbb{Z}$ denotes the integers, $\mathbb{Q}$ the rational numbers, $\mathbb{R}$ the real numbers, and $\mathbb{C}$ the complex numbers.

1. Let G be a finite group. Define the *Frattini subgroup* of G to be $\Phi(G)$, the intersection of all maximal subgroups of G.
   (1) Show that $\Phi(G)$ is characteristic in G (i.e. invariant under any automorphism of G).
   (2) Show that if $G = \langle \phi(G), S \rangle$ for some subset S of G, then $G = \langle S \rangle$.
   (3) Let P be a Sylow p-subgroup of $\phi(G)$. Show that P is normal in G (hint: first show that $G = \Phi(G)N_G(P)$ by using Sylow's theorems and then use (2)).
   (4) Show that $\Phi(G)$ is nilpotent.

2. Let G be a finite group acting on the finite set X with $|X| = n > 1$, and suppose that G has N orbits on X. If $g \in G$, let $F(g)$ be the number of $x \in X$ fixed by g.
   (1) Prove that $\sum_{g \in G} F(g) = |G|N$ (this is known as *Burnside's Lemma*).
   (2) Prove that if G is transitive on X, then $F(g) = 0$ for some $g \in G$ (either use (1) or prove directly).
   (3) Show that this is not always true if G is not transitive on X.

3. Let $f(x) = x^4 - x^3 + x^2 - x + 1 \in \mathbb{Q}[x]$. Find the splitting field (over $\mathbb{Q}$) of $f(x)$, and compute $Gal(K/\mathbb{Q})$.

4. Construct an example of each of the following (with reasons):
   (1) A field extension $F \subsetneq K$ which is normal but not separable.
   (2) A field extension $F \subsetneq K$ which is separable but not normal.
   (3) A field extension $F \subsetneq K$ which is neither separable nor normal.

5. Let F be the field of p elements. Let $A \in G := GL(n, F)$.
   (1) Show that A has order a power of p if and only if $(A - I)^n = 0$.
   (2) Show that if this is the case then the order of A is less than np.
   (3) Show that any such A is similar to an upper triangular matrix.

6. Let M be a finitely generated abelian group, and N a subgroup. If $M \otimes_{\mathbb{Z}} \mathbb{Q} \cong N \otimes_{\mathbb{Z}} \mathbb{Q}$, show that M/N is torsion.

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7. Consider the polynomial ring $\mathbb{C}[x, y]$ and let I be the ideal $I = (x + y - 2, x^2 + y^2 - 10)$.

(1) Show that there exists some $m > 0$ such that $(3x^2 + 10xy + 3y^2)^m \in I$.
(2) Show that the two ideals $I_1 = (x + y - 2)$ and $I_2 = (x^2 + y^2 - 10)$ are prime ideals. Are they maximal?
(3) Can I be written as an intersection of maximal ideals? Why or why not?

8. Let A be a finite-dimensional algebra over $\mathbb{R}$, with center $Z = Z(A)$ and Jacobson radical $J = J(A)$. Assume that for any $a \in A$, there is some $n = n(a) \geq 1$ such that $a^{2^n} - a \in Z$.

(1) Show that $J \subseteq Z$.
(2) Show that A/J is commutative.

In fact A itself is commutative, although you do not have to show this.