MATH 507a **QUALIFYING EXAM** **February 5, 2009**

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway.

1.) Assume $X, X_1, X_2, \ldots$ are independent identically distributed random variables, in the proper sense, having values in $(-\infty, \infty)$. Write $S_n = X_1 + \cdots + X_n$. The usual SLLN (strong law of large numbers) states that if $\mathbb{E}\, X = \mu \in (-\infty, \infty)$, then $S_n/n \to \mu$ almost surely. Use this, to prove the extended version: if $\mathbb{E}\, X = \mu \in [-\infty, \infty]$, then $S_n/n \to \mu$ almost surely.

2.) Let X_n be a sequence of finite, independent, nonnegative random variables such that $\lim_{n\to\infty} X_n = 0$ a.s. Prove that there is a non-random subsequence $n_1 < n_2 < n_3 < \ldots$ of the positive integers such that if

$$Y_m = X_{n_1} + X_{n_2} + \ldots + X_{n_m}$$

then

$$\lim_{m\to\infty} Y_m < \infty \text{ a.s.}$$

3.) Let X_1, X_2, ..., be a sequence of random variables. For every n, the density of X_n is given by

$$f_{X_n}(x) = \frac{\mathrm{sh}(x)}{n} \exp\left\{\frac{1-\mathrm{ch}(x)}{n}\right\} \mathbf{1}(x>0),$$

where $\mathrm{sh}(x) = (e^x - e^{-x})/2$ and $\mathrm{ch}(x) = (e^x + e^{-x})/2$. Prove or disprove the following statements, and <u>identify the limit</u> in the case when you claim that the convergence holds.

a)$\{\ln(\mathrm{ch}(X_n)) - \ln(n)\}_{n\geq 1}$, converges in law;
b) $\{\frac{\ln(\mathrm{ch}(X_n))}{\ln(n)}\}_{n\geq 1}$, converges in probability;
c)$\{\frac{X_n}{\ln(n)}\}_{n\geq 1}$, converges in probability.

4.) Let $X_1, X_2, \ldots$ be iid, with $E|X_1| < \infty$ finite and $EX_1 \neq 0$. Prove that

$$\max_{1\leq k\leq n} \frac{|X_k|}{|S_n|} \longrightarrow 0 \text{ a.s.}$$

Hint: First show that $\frac{|X_n|}{n} \to 0$ a.s.