Screening Exam in Numerical Analysis – Spring 2009

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1. Linear systems

Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 5 & 5 \\ 1 & 5 & 14 \end{pmatrix}$.

a. Compute LU decomposition of A, i.e. find such L and U that $A = LU$.

b. Show that A is a SPD matrix. Then compute Cholesky decomposition of A, i.e. find such L that $A = LL^T$.

2. Least Squares

Consider $Ax = b$, where $A \in R^{m\times n}$, $b \in R^m$. A minimum norm solution of the least squares problem is a vector $x \in R^n$ with minimum Euclidian norm that minimizes $\|Ax - b\|_2$.

a. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\quad b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

Find range and null space of A^T, find least squares solution to $Ax = b$, and find minimum norm solution: min $\|x\|_2$.

b. Show that a vector that minimizes $\|Ax - b\|_2$ is a minimum norm solution if and only if x is in the range of A^T.

3. Eigenvalue problems

a. Describe the QR iteration algorithm, present steps of efficient implementation, indicate why the method is numerically stable.

b. Verify that the eigenvalues are preserved in each step of shifted QR iteration algorithm.

c. What choice of the rotation angle θ will make A_0 tridiagonal?

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix} = U^{-1}AU,$$

where $s = \sin\theta$, $c = \cos\theta$, $|\theta| \leq \pi/2$.

4. Iterative methods

a. Consider the iterative method $x_{k+1} = -2x_k + b$ to solve the linear system $3Ix = b$, where I is $n \times n$ identity matrix.

For what values of the initial vectors x_0 the iteration converges? What is the spectral radius of iteration matrix?

b. Let A be a $n \times n$ matrix such that $A = (1 + \omega)P - (N + \omega P)$, with $P^{-1}N$ nonsingular and with real eigenvalues $1 > \lambda_1 \geq \ldots \geq \lambda_n$.

Find the values $\omega \in \mathbf{R}$ for which the following iterative method

$$(1 + \omega)Px_{k+1} = (N + \omega P)x_k + b,$$

with $k \geq 0$, converges to the solution of $Ax = b$ for every initial vector x_0.

Determine the values of ω for which the convergence rate is maximum.