Spring 2007 Math 541b Exam

1. Let $\mathbf{p} = (p_1, \ldots, p_c)$ be a vector of positive numbers summing to one, and $\mathbf{X} \sim \mathcal{M}(n, \mathbf{p})$, the multinomial distribution given by

$$P(\mathbf{X} = \mathbf{k}) = \binom{n}{\mathbf{k}} \mathbf{p}^{\mathbf{k}},$$

where $\mathbf{k} = (k_1, \ldots, k_c)$ are non-negative integers summing to n,

$$\binom{n}{\mathbf{k}} = \frac{n!}{\prod_{i=1}^{n} k_i!} \quad \text{and} \quad \mathbf{p}^{\mathbf{k}} = \prod_{i=1}^{n} p_i^{k_i}.$$

For a given probability vector $\mathbf{p}_0$ we test $H_0 : p = p_0$ versus $H_1 : p \neq p_0$ using the chi-squared test statistic

$$V^2 = \sum_{i=1}^{c} \frac{(X_i - np_{i,0})^2}{np_{i,0}}.$$

(a) Calculate the mean vector and the covariance matrix of $\mathbf{X}$.

(b) Define a matrix $\mathbf{P}$ such that

$$V^2 = n^{-1}(\mathbf{X} - n\mathbf{p})'\mathbf{P}^{-1}(\mathbf{X} - n\mathbf{p}).$$

(c) Show that

$$n^{-1/2}(\mathbf{X} - n\mathbf{p}) \to_p Y \sim \mathcal{N}_c(0, \Sigma)$$

(d) Find the distribution of $U = P^{-1/2}Y$, and show that the covariance matrix of U is a projection. (Recall that Q is a projection matrix if $Q' = Q^2 = Q$.) Hint: show

$$P^{-1/2}\Sigma P^{-1/2} = I - P^{-1/2}\mathbf{p}\mathbf{p}'P^{-1/2}.$$

(e) Show that

$$V^2 \to_d \chi^2_{c-1},$$

that is, that V^2 converges in distribution to a chi squared distribution with $c - 1$ degrees of freedom.

2. Suppose $X_1, \cdots, X_n$ are independently and identically distributed with variance σ^2.

(a) Show that the estimate of variance $\hat{\theta} = \sum_{i=1}^{n}(x_i - \bar{x})^2/n$ has bias equal to $-\sigma^2/n$ as an estimator of σ^2.

(b) Show that the bias of the jackknife estimate is $-s^2/n$, where $s^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2/n$.