Spring 2007 Math 541a Exam

1. Let $\mathcal{P} = \{y_1, y_2, \cdots, y_N\}$, where $y_i \in R$ are N distinct real numbers. The size N of $\mathcal{P}$ may be very large and it is impractical to sample all the values of $\mathcal{P}$. Suppose that we are interested in the population average

$$\mu = \frac{1}{N} \sum_{i=1}^{N} y_i.$$

In a survey, a subset $\mathbf{S} \subset \mathcal{P}$ of n elements, $0 < n < N$, are selected from $\mathcal{P}$ without replacement and the values are recorded as $X_1, X_2, \cdots, X_n$.

(a) Describe the probability space and the resulting joint distribution of (X_1, X_2).

(b) Calculate the mean EX_1 and the covariance $\text{Cov}(X_1, X_2)$.

(c) Suppose that we use the sampling average

$$\hat{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

to estimate the population mean μ. Show that $\hat{X}$ is an unbiased estimator of μ.

(d) Show that

$$\text{Var}(\hat{X}) = \frac{N-n}{n(N-1)} V_y,$$

where

$$V_y = \frac{1}{N} \sum_{j=1}^{N} (y_j - Y)^2.$$

2. Suppose that $X = (X_1, X_2, \cdots, X_n)$ follows a first order autoregressive model

$$X_t - \mu = \rho(X_{t-1} - \mu) + \epsilon_t, \qquad t = 1, 2, 3, \cdot, n,$$

where $\mu \in R$ and $\rho \in (-1, 1)$ and unknown and ϵ_t's are iid from $N(0, 1)$. Let $\theta = (\mu, \rho)$. Suppose $X_0 = 0$.

(a) What is the joint density function for $(X_1, X_2, \cdots, X_n)$.

(b) Find the maximum likelihood estimator for (μ, ρ).

(c) Calculate Fisher's information matrix and a lower bound for the variance of an unbiased estimator of μ^2.