Geometry/Topology Qualifying Exam

Spring 2007

Solve all **SIX** *problems. Partial credit will be given to partial solutions.*

1. (15 pts) Let $M_n(\mathbb{R})$ be the space of all $n \times n$ matrices with real entries. (This is, of course, a differentiable manifold.) For $A \in M_n(\mathbb{R})$, define a tangent vector to $M_n(\mathbb{R})$ at the identity matrix I to be the class of the curve $I + tA$, $-\epsilon < t < \epsilon$. Denote this tangent vector by $\vec{A}$.
   (a) For any $X \in M_n(\mathbb{R})$, let $R_X : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be defined by $R_X(B) = B \cdot X$. Prove that R_X is differentiable.
   (b) For any $\vec{A} \in T_I M_n(\mathbb{R})$, define a vector field $\xi_{\vec{A}}$ on $M_n(\mathbb{R})$ so that $\xi_{\vec{A}}(X) = (R_X)_*(I)(\vec{A})$. (Here $(R_X)_*(I)$ is the derivative of R_X at I.) Compute the Lie bracket $[\xi_{\vec{A}}, \xi_{\vec{B}}]$.

2. (15 pts) Let C be the subset of $\mathbb{C}^2$ with coordinates z, w, defined by the equation $w^2 = P(z)$, where $P(z)$ is a polynomial of degree 3.
   (a) Prove that if P has no repeated roots, then C is a submanifold of $\mathbb{C}^2$. (Remark: C is a complex submanifold, and hence is also a real submanifold.)
   (b) Suppose that P has no repeated roots. Compute the fundamental group of $C - \{(z, w) | w = 0\}$. (Hint: Think of covering spaces.)

3. (10 pts) Prove that the tangent bundle TM of a smooth manifold M has the structure of a smooth orientable manifold. (Do not assume that M itself is orientable.)

4. (10 pts) Consider the differential 1-form $\omega = dz - y\,dx$ on $\mathbb{R}^3$ with coordinates (x, y, z). Prove that $f\omega$ is not closed for any nowhere zero function $f : \mathbb{R}^3 \to \mathbb{R}$.

5. (10 pts) Define the notion of a *deformation retraction* of a space X onto a subset $A \subset X$. Prove that if A is the knot in the solid torus $X = S^1 \times D^2$ as drawn in the picture below, then there is no deformation retraction of X onto A.

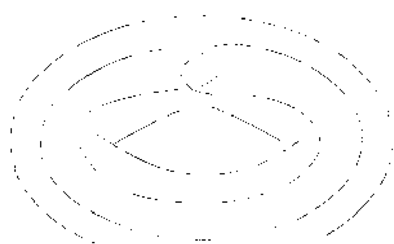

FIGURE 1

6. (10 pts) Construct a topological space X such that $H_0(X;\mathbb{Z}) = \mathbb{Z}$, $H_3(X;\mathbb{Z}) = \mathbb{Z}/5\mathbb{Z}$, $H_5(X;\mathbb{Z}) = \mathbb{Z}$, and all other homology groups are zero.