REAL ANALYSIS GRADUATE EXAM

SPRING 2007

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Let $\{\mu_n\}$ be a sequence of measures on $(X, \mathcal{M})$ with $\mu_1(E) \le \mu_2(E) \le \dots$ for all $E \in \mathcal{M}$. Let $\mu(E) = \lim_n \mu_n(E)$. Show that μ is a measure.

(2) Suppose $(X, \mathcal{M}, \mu)$ is a measure space with $\mu(X) < \infty$, and $f \in L^1(\mu)$ is strictly positive. Let $0 < \alpha < \mu(X)$.

(a) Show that

$$\inf \left\{ \int_E f \, d\mu : \mu(E) \ge \alpha \right\} > 0.$$

(b) Show that (a) can be false if we remove the assumption $\mu(X) < \infty$.

(3) Let $f : [0,1] \to \mathbb{R}$ be a continuous function. The *graph* of f is $\{(x, f(x)) : x \in [0,1]\}$. Show that the graph has two-dimensional Lebesgue measure 0.

(4) Let $n \ge 1$. Show that the function

$$g(u) = \int_{-\infty}^{\infty} \frac{x^n e^{ux}}{e^x + 1} \, dx, \qquad u \in (0,1),$$

is differentiable in $(0,1)$.