COMPLEX ANALYSIS GRADUATE EXAM

SPRING 2007

Answer all five questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. D denotes the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane.

Note that some problems are worth 10 points and others are worth 7 points.

1. (10 points) Let $\Omega \subset \mathbb{C}$ be a convex domain and let $f : \Omega \to \mathbb{C}$ be a nonconstant holomorphic function satisfying $\mathrm{Re}(f'(z)) \geq 0$ for all $z \in \Omega$. Prove that f is injective on Ω.

2. (10 points) Let f be an analytic function on D satisfying $|f(z)| \leq 1$ for all $z \in D$ and having at least two fixed points z_1 and z_2. Show that $f(z) = z$ for all $z \in D$.

3. (10 points) Find a conformal mapping of the semicircular region $R = \{z : \mathrm{Im}(z) > 0, |z| < 1\}$ onto D. HINT: You may decompose this map into simpler ones.

4. (7 points) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{e^{-ix}}{x^2 - 2x + 4} \, dx.$$

Justify your method.

5. (7 points) How many roots does the polynomial $p(z) = 2z^5 + 4z^2 + 1$ have in D? Justify your answer.