PROBABILITY (507a) QUALIFYING **EXAM** SPRING **2007**

Answer all three questions. Partial credit will be awarded, but in the event that you [illegible] fully solve a problem you should stat[illegible] what it [illegible] have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Suppose that $\{X_n : n \geq 1\}$ are independent identically distributed real valued random variables.

(i) Show that $X_n/n \to 0$ in probability.

(ii) Show that $X_n/n \to 0$ almost surely if and only if $E|X_1| < \infty$.

(iii) Find necessary and sufficient conditions for $X_n/\sqrt{n} \to 0$ almost surely.

2. (i) Suppose that X is an integer valued random variable with characteristic function $\phi_X(t)$, $t \in \mathbb{R}$. Show that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-itk} \phi_X(t)\,dt = P(X = k)$$

for all integers k.

(ii) Now suppose that $X_1, X_2, X_3, \ldots$ are i.i.d. with the same distribution as X and write $S_n = X_1 + X_2 + \ldots + X_n$. Find a similar integral formula for $P(S_n = k)$ in terms of the characteristic function ϕ_X.

3. Suppose that for each $n \geq 1$ the random variable X_n is normal with mean μ_n and standard deviation σ_n.

(i) Show that the family $\{X_n : n \geq 1\}$ is tight if and only if $\sup_n |\mu_n| < \infty$ and $\sup_n \sigma_n < \infty$.

(ii) Show that X_n converges in distribution to some [illegible] variable [illegible] if and [illegible] if there exist $\mu \in \mathbb{R}$ and $\sigma \in [0, \infty)$ such that $\mu_n \to \mu$ and $\sigma_n \to \sigma$.