**Screening Exam in Numerical Analysis, Spring 2007**

There are questions on 4 chapters covered in M502a.

I. *Linear systems*

1. Give a definition of a Symmetric Positive Definite (SPD) matrix $A \in R^{n\times n}$.

2. Show that no pivoting is necessary during Gaussian Elimination of a SPD matrix.

3. Show that matrix $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$ is SPD .

II. *Iterative solutions.*

Consider solving a system $Ax = b$, where $x, b \in R^n$, $A \in R^{n\times n}$ using semi–direct iterations, i.e. splitting the system into $Mx = (M - A)x + b$ and iterating as $Mx_{k+1} = (M - A)x_k + b$, or $x_{k+1} = Qx_k + c$, where the iteration matrix $Q = I - M^{-1}A$.

1. Define $M$ for the Jacobi, Gauss–Seidel and SOR iterations.

2. Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ .

a. Find $Q$ for the Jacobi, Gauss–Seidel and SOR iterations.

b. Find the optimal overrelaxation parameter $\omega$ for this $Q_{SOR}$.

c. Compute spectral radius of all three $Q$ matrices and compare their rate of convergence.

III. *Eigenvalue problems*
1. Prove the *Gerschgorin's theorem*: All the eigenvalues of the matrix $A \in C^{n\times n}$ lie in the union of the Gerschgorin disks in the complex plane

$$\mathcal{D}_i = \{z : |z - a_{ii}| \leq r_i\}, \quad r_i = \sum_{j\neq i, j=1}^{n} |a_{ij}|, i = 1, 2, \ldots, n.$$

Moreover, if the union $\mathcal{M}$ of $k$ Gershgorin disks $\mathcal{D}_i$ is disjoint from the remaining disks, then $\mathcal{M}$ contains precisely $k$ eigenvalues of $A$.
2. Consider the matrix

$$B = \begin{pmatrix} -5 & 1 & 0 & 0 \\ a & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

with $1 \leq a \leq 3$. Show that the dominant eigenvalue of $B$ is real.

IV. *Least squares*
1. Given the matrix $A \in \mathbb{R}^{m\times n}$ with rank $n$. Show that $A^T A$ is symmetric positive definite.
2. Suppose $b \in \mathbb{R}^m$. Show that $x = (A^T A)^{-1} A^T b$ minimizes $\|b - Ax\|_2$.
3. If the matrix is given by

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 0 \end{pmatrix}.$$

Find pseudoinverse $A^\dagger$.