REAL ANALYSIS GRADUATE EXAM

SPRING 2008

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let m denote Lebesgue measure on the unit square $E = [0,1] \times [0,1]$. In each case determine whether the integral exists:

$$\text{(i)} \quad \int_E \frac{1}{x-y}\, dm(x,y), \qquad \text{(ii)} \quad \int_E \frac{1}{x+y}\, dm(x,y).$$

2. Let f be a nonnegative measurable function on [0,1] such that $\int_0^1 f(x)\,dx = 1$. Define a measure μ on $[0,1]$ by

$$\mu(A) = \int_A f(x)\,dx, \quad A \in \mathcal{B}([0,1]).$$

Let K be the intersection of all compact subsets E of $[0,1]$ such that $\mu(E) = 1$. Find $\mu(K)$.

3. For a function $f : [0,1] \to \mathbb{R}$ determine whether either of the statements

"f is continuous almost everywhere on $[0,1]$"

and

"there is a continuous $g : [0,1] \to \mathbb{R}$ such that $f = g$ almost everywhere"

implies the other one. In each case justify your answer with a proof or counterexample.

4. Suppose that $f \in L^1(\mathbb{R})$. Prove that

$$\lim_{m\to\infty} \sum_{k=-m^2}^{m^2} \left| \int_{k/m}^{(k+1)/m} f(x)\,dx \right| = \|f\|_{L^1(\mathbb{R})}.$$