REAL ANALYSIS GRADUATE EXAM

FALL 2007

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. In questions 3 and 4, $|A|$ is used to denote the Lebesgue measure of a measurable subset $A \subseteq \mathbb{R}^d$.

1. By differentiating the equation

$$\int_{-\infty}^{\infty} e^{-tx^2}\,dx = \sqrt{\frac{\pi}{t}} \qquad t > 0$$

show that

$$\int_{-\infty}^{\infty} x^{2n} e^{-x^2}\,dx = \frac{(2n)!\sqrt{\pi}}{4^n n!}$$

for $n \geq 1$. You should be careful to justify your calculations.

2. (a) Construct a sequence f_n of Lebesgue measurable functions on $(0,1)$ such that $f_n(x) \to 0$ as $n \to \infty$ for each $x \in (0,1)$ and

$$\int_0^1 |f_n(x)|\,dx \to \infty$$

as $n \to \infty$.

(b) Give an example of a continuous function $F : [a,b] \to \mathbb{R}$ which is differentiable almost everywhere in $[a,b]$ with F' Lebesgue integrable on $[a,b]$ and such that

$$F(b) - F(a) \neq \int_a^b F'(t)\,dt.$$

3. Let g_k, $k = 1, 2, \ldots$, be a sequence of nonnegative measurable functions on a measurable subset E of $\mathbb{R}^d$. Suppose that

$$|\{x \in E : g_k(x) > 1/2^k\}| < 1/2^k \quad \text{for each } k \geq 1.$$

Prove that $\sum_{k=1}^{\infty} g_k$ converges almost everywhere on E .

4. Let $g \in L^p(\mathbb{R}^d)$ and define

$$\mu(t) = |\{x \in \mathbb{R}^d : |g(x)| > t\}| \quad \text{for } t \geq 0.$$

Show that

$$\int_{\mathbf{R}^d} |g(x)|^p\,dx = -\int_0^{\infty} t^p\,d\mu(t).$$