REAL ANALYSIS GRADUATE EXAM
Fall 2008

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Let μ, ν be finite Borel measures on $\mathbb{R}^2$ such that $\mu(B) = \nu(B)$ for every open triangular region B in the plane. Show that $\mu(E) = \nu(E)$ for all Borel sets E. [Note added later: This is a modified version of the problem actually asked, which was inappropriately difficult, with balls in place of triangles.]

(2) Show that

$$\lim_{n\to\infty} \int_0^\infty \frac{1 + nx^2 + n^2x^4}{(1+x^2)^n}\, dx$$

exists, and determine its value.

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function and let m be Lebesgue measure. Suppose there exist $M > 0$ and $c \in (0,1)$ such that

$$m\big(\{x : |f(x)| \geq t\}\big) \leq \frac{M}{t^c} \quad \text{for all } t > 0.$$

Show that f is Lebesgue integrable.

(4) Let $T_0^1(g) = \sup_{a=x_0<x_1<\cdots<x_n=b} \sum_{i=1}^n |g(x_i) - g(x_{i-1})|$ denote the total variation of a function $g : [0,1] \to \mathbb{R}$. Suppose that f_n, f are real-valued with $f_n(x) \to f(x)$ for all $x \in [0,1]$.

(i) Show that $T_0^1(f) \leq \liminf_{n\to\infty} T_0^1(f_n)$.

(ii) If we also assume each f_n is absolutely continuous and $T_0^1(f_n) \leq 1$ for all n, is it necessarily true that $T_0^1(f) = \lim_{n\to\infty} T_0^1(f_n)$? Justify.